

Mathematical modelling and numerical simulation of coupled acoustic multi-layer systems for enabling particle velocity measurements in the presence of airflow

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PhD thesis, University of A Coruña / 2021

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PhD program in Mathematical Modelling and Numerical Simulation in
Engineering and Applied Science



UNIVERSIDADE DA CORUÑA

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Mathematical modelling and numerical simulation of coupled acoustic multi-layer systems for enabling particle velocity measurements in the presence of airflow

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This thesis is dedicated to my parents.

*My Dad, who did everything and beyond to educate
and encourage me to go on my very own pursuits.*

*To Mom, who has been a source of warmth
and affection all my life.*

Acknowledgements

This thesis is a reality solely because of my supervisor, Dr. Andrés Prieto. It has sincerely been a pleasure working with him. He has been an overwhelming source of guidance, inspiration and support - be it academic or otherwise and I aspire to inculcate not just his particular devotion to his research, but also the philosophy he approaches life in general. He has been a friend, a mentor and a *guru*, and I would like to express him my deepest gratitude.

Next I would like to thank my industrial supervisor, Dr Daniel Fernández Comesaña. He has been instrumental and proactive in offering his suggestions and help whenever asked for. I am indebted to him for all the encouragement during the course of the project and also helping me settle at ease in the Netherlands.

I would also like to take this opportunity to also thank the director of my host institution, ITMATI and my mentor in the ROMSOC project, Dr. Peregrina Quintela. Additionally, I would like to thank Dr Elena Vázquez, Dr Fernando Varas (Curro), Dr Elena Martín and the faculty of this inter-university doctoral program. They have all been nothing but warm and helpful in every regard and it was pleasure to interact with them.

I would also like to thank all the friends I have made in Spain - Branca, Sebastian, Rubèn, Adriana, Ariana, Raluca, for making me feel right at home and bearing with me during my stay here in Spain especially due to my inability to speak Spanish. Their support has been invaluable during the past few years.

Similarly - I am grateful for all the colleagues at Microflown and friends in the Netherlands - Enzo, Fanyu, Krishna, Grachi, Mattheus, Ashwin (Fernandes) and Marcus for sharing moments of fun and joy during my stay in Netherlands and extending help when necessary. It is unthinkable how sombre it would have been without them amidst the COVID-19 pandemic.

For the ESRs in our project - I am grateful for the banter that we have had and the friendship we have grown in our unique doctoral journey.

Many people have been instrumental in my journey towards my PhD. I would like to particularly thank Dr. Swetaprovo Chaudhuri and all the members of his lab, for giving me an initial opportunity to explore research. Dr. K.N Lakshmisha (IISc), Dr Michael Holst (UCSD), Dr. Julius Kuti(UCSD) - have all encouraged me with their support.

Finally, it is impossible for me to express the support I have had from my family, friends and teachers along the years. It would be injustice to name just a few - and I am absolutely indebted to them.

I am incredibly grateful to my parents - I am incredibly blessed to receive your love, caring and encouragement. Also I express my thanks to my sister, my in-laws and long-time friends have always been my solace to seek conversation and motivation. Finally I am really grateful to all the understanding, love and caring I have gotten from my wife. She has been entirely instrumental during my entire doctoral journey involving short-term moving between the countries and late-night deliberations.

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Preface

Microflown PU probes, which are able to measure particle velocity and acoustic pressure fields simultaneously, are sensitive to the effect of wind, since they are based on transducers that are highly dependent on the variations of flow velocity. Objectives of this research project are the mathematical modelling and numerical simulation of thermo-acoustic coupled systems (involving PU probes, the compressible fluid in the presence of flow, and the multilayer windscreen). The numerical results will play a key role in the design of novel windscreens to mitigate the flow effects on the measurements of acoustic probes.

The state-of-the-art sensor designed by Microflown Technologies is the so called USP; "the Ultimate Sound Probe" [72]. The three dimensional $\frac{1}{2}$ inch USP regular probe consists of a three orthogonally placed acoustic particle velocity sensors and a sound pressure microphone. The actual sensor configuration without its cap is less than $5 \times 5 \times 5 \text{ mm}^3$. Any sound field is described by two complementary acoustic properties, the scalar value 'sound pressure' and the vector value 'particle velocity'. Currently, the USP probe is the only sensor which can physically measure the acoustic particle velocity directly. The USP regular probe is mainly used as an AVS (Acoustic Vector Sensor). More precisely, a single AVS device can be used for source localization and target tracking. The most important feature of AVS is its small size and low weight, which can be less than 1 cm^3 and 100 grams, respectively, which makes it more attractive for industrial applications than the traditional sensor array [30]. Beneficial of the measurement principle is that measurement can be done in-situ under realistic condition. The sensors have no need to create anechoic conditions and can be used within closed cavities, such as a car interior, the sensor is not having P/I (pressure over intensity) index problems.

The effect of wind is considerable on a Microflown transducer since it is sensitive to variations of air flow. Three main effects have been thought to influence the response: (a) Since the Microflown transducer is mainly a thermal device, the probe wires have to be heated for proper operation. If the mean flow is too high, the sensor wires cool down and the device stops working; (b) Even if the mean flow is blocked properly, it is still possible that the low frequency variation in the mean flow is so large that the pre-amplifier overloads and that the amplified signal starts clipping. At higher wind speeds, a physical obstacle causes turbulences that translates into acoustic noise; (c) In general frameworks, the self noise induced by the flow is the lower limit when the radiated noise has to be measured. Only if the source acoustic signal is known a priori, the acoustic signal can be used to measure below the self noise. Hence, there is a need to consider different windscreen strategies with

the aim of mitigating the potential influence of airflow on the probe measurements.

The project focuses on mathematical modelling and numerical simulation of a coupled system consisting of: the probe, the compressible fluid in the presence of flow (where the interaction of fluid-structure has to take into account the computation of a turbulent or laminar flow around the probe), and the multi-layer structure that acts as a windscreen (which, depending on its nature, could be modelled as a multilayer poroelastic material or a thin structural component with different geometric scales). Since the effects of the airflow are mainly visible at a lower frequency range, the application of multiscale techniques [43] are adequate to compute accurate numerical results and subsequently, to optimize the design parameters of the windscreen device.

The present project is mainly focused on the work package 2 “Coupled problems” within the Reduced Order Modelling, Simulation Optimization of Coupled Systems (ROMSOC) project. More precisely, since the wind-shield enclosures generate complex acoustics fields inside [60], three stages are considered by solving numerically the following coupled problems: (A) Modelling and numerical simulation of the three dimensional coupled acoustic behaviour of a multilayer structure composed of metallic micro-perforated screens surrounding the PU probe. The characterization of the mechanical impedance of this protected screen will be computed in the presence of flow. (B) Modelling and numerical simulation of the three dimensional coupled acoustic behaviour of a multilayer system consisting of open porous foams. (C) Modelling and numerical simulation of the three dimensional coupled acoustic behaviour of airflow around the multilayer windscreens

Eventually, once the performance of the different windscreen strategies have been evaluated numerically, it is to be considered in combination. All the numerical simulations are performed by using open software packages in accordance with the spirit of open-access research. Towards this end, A brief description of each chapter of the thesis is presented below:

In Chapter 1, we introduce the ROMSOC project, the Microflow sensor and highlight the difficulties in performing measurements in presence of flow. More importantly, We develop a staggered approach to develop coupled mathematical models to quantify the acoustic behavior of the probe housing and the materials within a windscreen.

In Chapter 2, different strategies to compute pressure projection fields from displacement field solutions are investigated for unstructured grids. Numerical results are compared to the accuracy of the projection techniques

Chapter 3 deals with the modeling and numerical simulation of package gain and insertion loss due to porous and steel meshes on acoustic probes in presence of flow. A fluid equivalent porous model and an impedance interface model is suggested for the steel mesh and subsequently coupled using Raviart-Thomas finite elements. Numerical results are reported for the coupled simulation and its influence on sound probes.

Chapter 4 takes a structured approach from continuum mechanics basics to develop a mathematical model for acoustic propagation in porous materials in presence of an underlying flow. We introduce the Galbrun’s model for modelling acoustic perturbations in presence of flow and extend it for fluid-equivalent poroelastic material called the Darcy-

Forcheimer-Galbrun model.

Chapter 5 highlights the numerical simulations for computing acoustic propagation for the models suggested in Chapter 4. It implements a regularized Galbrun's equation model and describes a series of validation tests conducted before reporting the results with the PU probe. Moreover, it also shows some numerical simulations with a simplified version of the Darcy-Forcheimer-Galbrun models.

Chapter 1

Introduction

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1.1 ROMSOC Project

The Reduced Order Methods, Simulation, Optimization and Coupled Systems (ROMSOC) project is a European Industrial Doctorate (EID) program running between 2017-2021, bringing together 15 international academic institutions and 11 industry partners within Europe including University of ACoruña. The European Commission sponsors the European Industrial Doctorate Programs (EIDs) within the framework of Horizon-2020 as a type of "Innovative Training Networks" project, a part within the Marie Skłodowska-Curie Actions (MSCA) supporting the careers of scientists and encouraging their transnational, intersectoral and interdisciplinary mobility.

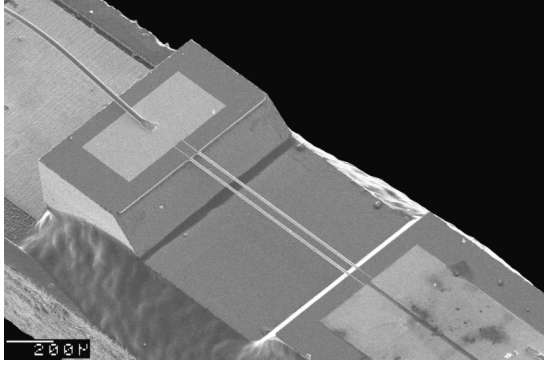
The ROMSOC project aims to address the key element in the dawn of a new digital industrial era - mathematical models serving as digital twins of the real products and processes forming the basis for optimization and control of design and functionality. It aims to achieve the best performance of modeling, simulation, and optimization techniques (MSO), with particular focus on industrial applications. Towards these goals, it supports the training of eleven Early-Stage-Researchers (ESRs) in a multi-disciplinary, international doctoral program mentored by a European university and an industry partner to address their scientific training and soft skills. The doctoral program includes a mandatory requirement of thorough training across its partner universities across Europe and for the researcher to exercise international mobility and spend at least half the duration of the program within an industrial partner in a different country.

The current thesis is a result of one of the eleven sub-projects under ROMSOC under a collaboration between University of ACoruña, Spain through its partnering research laboratory, Instituto Tecnológico Matemático Industrial (ITMATI) and Microflow Technologies B.V., Netherlands. Under the broader goals set by the ROMSOC project, this sub-project particularly focuses on developing coupled mathematical models and numerical simulations in acoustics. Objectives include developing coupled systems for improving the measurements of particle velocity sensors by modelling multi-layer windscreens in presence of airflow. The industry partner, Microflow Technologies is a manufacturer of particle-velocity sensors aptly named *the Microflow* and the developed models are utilized to investigate the impact of porous windscreens and laminar flow on their particle velocity probes.

Abiding by the spirit of open-access research policy of the Horizon-2020 framework, this project provides access, information and training to utilize the research data, software and publications. Towards this goal, this research uses only open-access and open-source tools to demonstrate the results and an appendix is also included within the thesis pointing towards the project software repository while also describing use cases and implementation.

1.2 Introduction

The Microflow PU probes are acoustical sensors distinguished by their ability to measure fluctuations in particle velocity and pressure fields simultaneously and are of consider-



(a)



(b)

Figure 1.1: (a) SEM image of the Microflow's particle velocity transducer, and (b) the PU Probe housing the particle velocity and pressure sensors

able importance in sound measurement. They have been used to measure sound-intensity, acoustic impedance, pressure and have been adopted for a variety of applications like sound localization, acoustic quantification and noise control. In the past 24 years since its invention at University of Twente unto present, it is said to be the only sensor to measure particle velocity directly off the acoustic field.

The sensor works on the principal of differential heating. The microscopic transducer (Fig1.1a) consists of a two tiny, resistive strips of platinum which measure the acoustic particle velocity. In the presence of particle velocity, the temperature distribution around the resistors is asymmetrically altered, and a difference in temperature occurs between the two wires, as the upstream wire is cooled more than the downstream wire by the acoustic airflow. The resulting resistance difference translates into variations in a signal which captures the particle velocity of a single point of a sound field in an arbitrary direction.

The Microflow sensor being used in different environmental conditions, is observed to be sensitive to flow conditions. Since the transducer is mainly a thermal device, the probe wires have to be heated for proper operation. Previous models of the microflow transducers [68], [71] indicate that at large flow velocities, the metal wires, having a finite heat capacity, cools down and the signal can distort considerably or die down unexpectedly. Due to this physical limitation, the sensor is usually used with a windscreen when measurements are taken in an environment which has a significant air flow. The company offers multiple windscreens in various sizes and materials, from cylindrical porous covers to metallic meshes.

For measurements in presence of no flow, studies indicate that introduction of wind-screens on the sensor induce significant insertion loss and phase error in the particle velocity measurements [66],[37]. While in presence of flow, windscreen induced flow resistance can also lead to low frequency variations in sound leading to loss in accuracy when amplified. At higher flow velocities, any introduction of physical obstacle causes additional turbulence scales which effectively translates to acoustic noise. Moreover, since the self-noise induced by the flow is used as reference to which the acoustic noise is measured, a dynamic calibra-

tion is required in such cases. While these problems can complicate the matter of utilizing windscreens for accuracy, they are indispensable for effective measurements. Hence, different wind-screening strategies should be considered with the aim of mitigating such effects on the probe measurements.

The objective of this project is to study the effects of introducing a porous media layer over a sound intensity PU probe in presence of flow to prevent noise inhibitions. A computational model simulating the behaviour of the sensor can highlight the physical characteristics and help optimize the design for improving the measurement. This report shows the approach towards development of such a model and highlights the numerical results to measure the impact of the windscreen of the sensor on its measurement.

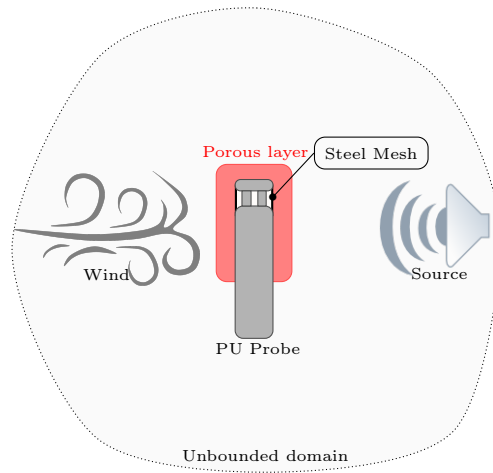


Figure 1.2: A schematic of the objective coupled problem

1.3 Model development

The physical behaviour of the acoustical sensor enclosed by a porous material in flow requires an astute understanding of sound propagation over the porous structure in different flow conditions. The material properties and geometry of the porous windscreen can either dampen or accentuate either the magnitude or the phase of the measured signal. Additionally, turbulence in the acoustic fluid can give rise to noise at various frequency ranges. In order to predict these responses accurately it is necessary to model the coupling between the various physical models of flow and the involved structures.

A series of hierarchical models and associated assumptions are considered to arrive at a feasible overall numerical solution for the problem. We consider the development of a computational model to simulate such coupled behaviour in 6 stages as shown in table 1.1. Each stage is represented by a schematic diagram indicating the coupling required. For simplicity, we begin the model development with a probe in free-field conditions with fluid at rest and an incident acoustic wave. The model plainly describes the effect of package

induced signal gain of the probe. This is easily quantified by comparing the signal characteristics with the known design documents. The model is further developed to incorporate the coupling with the steel mesh to help understand the acoustic behaviour in presence of such material.

To study the effects of a windscreen on the acoustic fluid, we start with coupling the initial model with the porous windscreen. Subsequently a joint coupled model is developed with the steel mesh and the porous layer.

To study the effects of flow on the acoustic fluid, we then start with coupling the initial model with the fluid assumed to be laminar and non-viscous. This model is also later evolved to incorporate a porous housing. Since an effective coupled model to compute sound propagation in flow in presence of porous materials does not exist, a model is developed from basics.

Each stage of the mathematical model developed can be tested and validated by comparing the numerical results obtained to experimental data obtained from the company. Figure 1.1 lists each of the six cases along with the schematic representation. Once the mathematical model is developed for a simplified geometry in no-flow conditions, the same model may be applied to the probe housing to study the theoretical effect of its presence in the measurement. The numerical results obtained can be verified experimentally to validate the results. The model is then developed to couple with the material properties of a porous windscreen, to arrive at the predicted difference in gain of the particle velocity signal. This can again, be corroborated by reproducing the experiment in a laboratory setup.

Adding flow conditions to the initial model will require re-examining the basic conservation principles and arrive at a different mathematical model. As before, it can be used to obtain an estimate of difference gain and phase response of the signal before venturing into complex flow experiments. We begin with the simpler laminar flow models at lower Reynolds number. Once validated, we proceed to couple it with the material response of the porous windscreen of the sensor. The methodology is developed over the course of the project.

The developed mathematical model is solved using discrete numerical techniques namely, the Finite Element method to arrive at a feasible solution. The problem can be mathematically formulated in various physically-relevant variables e.g scalar fields like pressure, displacement potential or velocity potential; or vector fields like displacement or velocity, the choice often being the vector fields for coupled systems [21] Formulations in pressure or scalar potential fields can lead to numerical issues due to the complicated contact conditions at the interfaces of the media. Hence this thesis uses the displacement formulation of the the governing equations extensively.

1.4 Original contributions

The main, original contributions of this thesis are listed below:

- The first contribution is a numerical comparison between post-processing strategies to

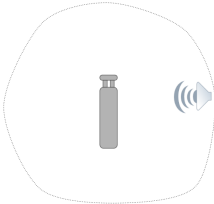
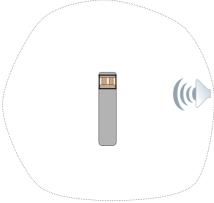
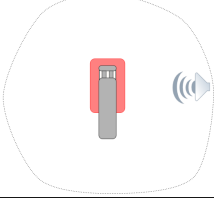
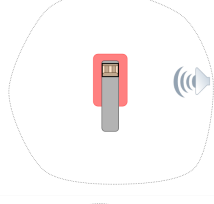
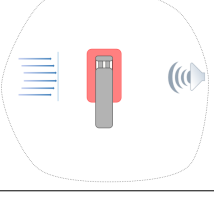
Case	Description	Schematic
1	Mathematical model of acoustics in fluid at rest	
2	Coupled acoustic model with steel mesh and fluid at rest	
3	Coupled model of acoustics with porous layer in still fluid	
4	Coupled acoustic model with steel mesh and porous windscreens	
5	Mathematical model of acoustic field in laminar flow	
6	Coupled model with porous windscreen in laminar flow	

Table 1.1: Benchmark Cases

compute the scalar pressure fields in displacement based models on an unstructured grid. Computing these indirectly derived physical quantities can provide valuable information in mechanical and vibro-acoustic problems and hence the accuracy of these techniques are discussed in detail by comparing against two testcase problems with exact solutions.

- Development of a coupled mathematical model and the methodology to quantify the acoustic insertion loss of multi-layer windscreen containing steel meshes and porous windscreens on particle-velocity sensors.
- Development of a numerical simulation solver suitable for evaluating the performance of an arbitrarily shaped windscreen suitable for sound pressure, particle velocity and sound intensity probes.
- A novel methodology which blends experimental techniques to characterize material properties of porous materials for use in coupled poro-acoustic models. The approach utilizes an *insitu* measurement of surface properties of porous materials to extract porous material parameters by means of an inverse model and utilizes this to gain insights into the sound field using computational methods. This provides for a more refined tool as against the traditional parameter-sweeping approach followed in acoustic simulations of porous materials.
- A rigorous development of a novel Darcy-Forcheimer-Galbrun model for acoustic propagation through porous materials in presence of laminar flows. The technique effectively decouples the models for underlying flow and acoustic propagation thereby paving way for a modular approach to poro-aeroacoustic simulations.
- A first implementation of coupled mathematical model to predict acoustic impact of laminar flow on particle velocity sensors. This is further extended to include coupling with porous windscreens utilizing a simplified Darcy-Forcheimer-Galbrun model.

Chapter 2

Numerical comparison of different projection strategies to compute the acoustic pressure field from displacement-based formulations

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2.1 Introduction

Sound travelling within an acoustic medium carries information about the source(s) and the characteristics of the medium. Real-world applications like geophysics, underwater scattering, automobiles and shipping industries utilise these properties to quantify relevant physical phenomena of interest. This is made possible by effective mathematical models of sound transmission within each layered media, a coupling hierarchy of models and a subsequent solver based on computational techniques such as finite element methods.

Models for sound propagation in acoustic fluids are commonly expressed in terms of the fluid pressure. Pressure being a scalar quantity is also a sensible choice in lieu of experiments. But for coupled problems, like fluid-structure interactions or layered media propagation, the models are more easily expressed in terms of the vector displacement field encouraged by the compatibility of boundary and equilibrium conditions at interfaces. The displacement formulation is known in literature to suffer from spurious modes when discretized with standard finite elements and mixed formulations were used in coupled problems until Bermúdez and Rodríguez [22] demonstrated the stability of displacement fields with Raviart-Thomas finite elements (see, for instance [34] for further details). This article follows the displacement formulation, in view of its coupled-media applications. To ensure a well-posedness of the problem, it is relevant to state the conditions at interface of two media and also at the boundaries of the system. In heterogeneous media, these are often expressed in terms of measurable quantities like the surface impedance or reflection, computed as the ratio of acoustical pressure and the particle velocity at the interface. For a coupled problem with the displacement field solution, it is hence required to compute of scalar acoustic pressure.

This requirement is all the more relevant with the advent of new data-driven modelling techniques which utilise measured quantities to improve the solution. While there are a variety of techniques to compute the pressure from the solution of the displacement field, a comparison of accuracy of these techniques is not found in literature. This article is focused on the detailed comparison of different post-processing techniques to compute pressure values from the displacement field in a variety of acoustic scattering problems.

2.2 Statement of the problem

Since acoustic scattering problems are often stated in unbounded domains, they involve radiation boundary conditions. In the present chapter, to mimic the outgoing behaviour of the scattered acoustic fields, the Perfectly Matched Layers will be used.

2.2.1 Scattering problem

Throughout this chapter, we focus on the time-harmonic wave propagation problem in isentropic and inviscid homogeneous compressible fluids, that is, their mechanical behaviour is characterised by a constant mass density ρ_0 and a constant sound speed c_0 . In absence

of volume excitation terms, the balance of forces along with the constitutive law for this type of compressible fluids lead to the following governing equations,

$$-\omega^2 \rho_0 \mathbf{u} - \text{grad } p = \mathbf{0}, \quad (2.2.1)$$

$$p = -\rho_0 c_0^2 \text{div } \mathbf{u}, \quad (2.2.2)$$

where ω is the angular frequency, $\mathbf{u}$ denotes the particle displacement field and p is the Lagrangian fluctuation of the pressure field (the so-called acoustic pressure). On one hand, if the pressure field is substituted in (2.2.2) and using ρ_0 and c_0 are non-null constant, we obtain the typical scalar Helmholtz equation

$$-\Delta p - k^2 p = 0, \quad (2.2.3)$$

where $k = \omega/c$ is the wavenumber. On the other hand, if we are focused on writing the governing equation of motion in terms of the displacement field $\mathbf{u}$, we compute the gradient of (2.2.2) and then the expression $\text{grad } p$ is substituted from (2.2.1). In that manner, we get the vector-valued version of the Helmholtz equation in terms of the displacement field, i.e.,

$$-\text{grad div } \mathbf{u} - k^2 \mathbf{u} = \mathbf{0}. \quad (2.2.4)$$

Both partial differential equations are linear and they model equivalently the time-harmonic vibrations of a compressible fluid. In the present work, we will focus on the use of (2.2.4) to compute the displacement field and then how to compute efficiently the pressure field as a post-processing step.

The scattering problems, where (2.2.4) holds, are stated in an unbounded open regular domain $\Omega \subset \mathbb{R}^d$, with $1 \leq d \leq 3$ with piece-wise smooth boundary. More precisely, the boundary $\partial\Omega$ is split in three disjoint parts Γ_D and Γ_N where different hard-wall and soft-wall boundary conditions are imposed, respectively. Hard-wall conditions are related with fixed boundaries where the normal displacement is prescribed (also called essential or Dirichlet boundary conditions for the normal displacement field). Otherwise, soft-wall conditions impose the value of the pressure field on the boundary (also called natural or Neumann boundary conditions since they are involving the evaluation of derivatives on the boundary).

Finally, the well-known Sommerfeld radiation condition must be included in the statement of the problem to preserve that only the outgoing time-harmonic scattered fields are computed. In the case of the classical scalar Helmholtz equation, the Sommerfeld radiation condition written in terms of the pressure field is given by

$$\lim_{|\mathbf{x}| \rightarrow \infty} |\mathbf{x}|^{(d-1)/2} \left(\text{grad } p \cdot \frac{\mathbf{x}}{|\mathbf{x}|} - ikp \right) = 0, \quad (2.2.5)$$

whereas, if it is rewritten in terms of the displacement field using (2.2.1) and (2.2.2), then we have

$$\lim_{|\mathbf{x}| \rightarrow \infty} |\mathbf{x}|^{(d-1)/2} \left(\text{div } \mathbf{u} - ik \mathbf{u} \cdot \frac{\mathbf{x}}{|\mathbf{x}|} \right) = 0. \quad (2.2.6)$$

In conclusion, the scattering problem can be stated in terms of the displacement field as follows: For a given wavenumber k , and boundary data u_D and p_N , find $\mathbf{u}$ such that

$$-\operatorname{grad} \operatorname{div} \mathbf{u} - k^2 \mathbf{u} = \mathbf{0} \quad \text{in } \Omega, \quad (2.2.7)$$

$$\mathbf{u} \cdot \mathbf{n} = u_D \quad \text{on } \Gamma_D, \quad (2.2.8)$$

$$-\rho_0 c_0^2 \operatorname{div} \mathbf{u} = p_N \quad \text{on } \Gamma_N, \quad (2.2.9)$$

$$\lim_{|\mathbf{x}| \rightarrow \infty} |\mathbf{x}|^{(d-1)/2} \left(\operatorname{div} \mathbf{u} - ik \mathbf{u} \cdot \frac{\mathbf{x}}{|\mathbf{x}|} \right) = 0, \quad (2.2.10)$$

where $\mathbf{n}$ is the unit vector normal to $\partial\Omega$ and outward with respect to Ω , and u_D and p_N are the normal displacement and the pressure boundary data on Γ_D and Γ_N , respectively. Due to the outgoing character of the radiation boundary condition, there exists a unique solution for the acoustic scattering problem (2.2.7)-(2.2.10).

2.2.2 Perfectly Matched Layers

The scattering acoustic problems are typically stated in unbounded domains. Hence, this domain should be truncated before the use of any discretization, since they require for a bounded computational domain. Obviously, the truncation of the original unbounded domain at a finite distance from the scatter and imposing some homogeneous Dirichlet of boundary conditions would introduce spurious reflections on the scattering problem (or even worse, it would generate artificial inner resonances which are not presented in the original scattering problem).

In the present work, the unbounded domain is truncated by surrounding the physical region of interest with an artificial peripheral damping layer which absorbs all outgoing waves. The design of this sponge layer is based on the so-called Perfectly Matched Layer (PML). The main idea of this truncation method consists in stretching [33] the spatial coordinates into a complex-valued spatial coordinates,

$$(x_1, x_2, x_3) \in \mathbb{R}^3 \quad \text{are replaced by} \quad (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in \mathbb{C}^3,$$

in such a manner that the outgoing waves, which are solutions of the scattering problem, will be damped in the interior of the PML domain. Different types of PML models can be written attending to the chosen system of coordinates (Cartesian, cylindrical, spherical, general convex systems). For instance, in the Cartesian PML model the usual spatial Cartesian coordinates are replaced by

$$\tilde{x}_j(x) = x_j - \frac{i}{k} \int_{L_j}^{x_j} \sigma_{x_j}(s) ds, \quad 1 \leq j \leq 3,$$

where $x_j = L_j$ is the coordinate plane where the common interface boundary between the PML layer and the physical domain of interest is placed, and σ_{x_j} is the PML absorbing profile, which determines the exponential decay of the outgoing waves inside the PML layer in the x_j direction, $1 \leq j \leq 3$.

In the case of the spherical PML model, only the radial coordinate r is complex-valued stretched to compute $\tilde{r}(r)$, which is given by

$$\tilde{r}(r) = r - \frac{i}{k} \int_R^r \sigma_r(s) ds, \quad (2.2.11)$$

being $r = R$ the sphere where the common interface boundary between the PML layer and the physical domain of interest is placed and σ_r is the PML absorbing profile in the radial direction. In this case of spherical PML layers, the Cartesian coordinates (x_1, x_2, x_3) are also complex-valued stretched and computed replacing r by $\tilde{r}(r)$ from the spherical coordinates (r, θ, ϕ) as follows:

$$\tilde{x}_1 = \tilde{r}(r) \cos \theta \sin \phi, \quad (2.2.12)$$

$$\tilde{x}_2 = \tilde{r}(r) \sin \theta \sin \phi, \quad (2.2.13)$$

$$\tilde{x}_3 = \tilde{r}(r) \cos \phi. \quad (2.2.14)$$

To avoid also spurious reflections from the inner interface between the physical domain of interest and the PML layer, the differential operators involved in the governing equations of motion must be modified accordingly. In that manner, the motion equation of the Cartesian PML layer associated with (2.2.4) is written by replacing formally the gradient of a scalar field, $\text{grad } \phi$, and divergence of a vector field, $\text{grad } \mathbf{w}$, by

$$\begin{aligned} \widetilde{\text{grad}} \phi &= \sum_{j=1}^3 \frac{\partial \phi}{\partial \tilde{x}_j} \mathbf{e}_j = \sum_{j=1}^3 \frac{1}{\gamma_j} \frac{\partial \phi}{\partial x_j} \mathbf{e}_j = H \text{grad } \phi, \\ \widetilde{\text{div}} \mathbf{w} &= \sum_{j=1}^3 \frac{\partial w_j}{\partial \tilde{x}_j} = \sum_{j=1}^3 \frac{1}{\gamma_j} \frac{\partial w_j}{\partial x_j} = H^{-1} : \text{grad } \mathbf{w}, \end{aligned}$$

where H is the Jacobian matrix associated to the complex-stretching of spatial coordinates, this is, it is the second order diagonal tensor defined by

$$H = \sum_{j=1}^3 \gamma_j \mathbf{e}_j \otimes \mathbf{e}_j, \quad \gamma_j(x_j) = \frac{d\tilde{x}_j}{dx_j}(x_j) \quad 1 \leq j \leq 3. \quad (2.2.15)$$

An analogous writing of the differential operators could be made for the spherical PML model replacing r by $\tilde{r}$ in the differential operators and defining H by

$$H = \sum_{i,j=1}^3 \gamma_{ij} \mathbf{e}_i \otimes \mathbf{e}_j, \quad \gamma_{ij}(x_1, x_2, x_3) = \frac{d\tilde{x}_i}{dx_j}(x_1, x_2, x_3) \quad 1 \leq i, j \leq 3. \quad (2.2.16)$$

where $\tilde{x}_j$ is given from (2.2.12)-(2.2.14) in combination with (2.2.11).

Consequently, if $\tilde{\mathbf{u}}$ denotes the displacement field in the PML layer, its governing equation is given by

$$-\widetilde{\text{grad}} \widetilde{\text{div}} \mathbf{u} - k^2 \mathbf{u} = \mathbf{0},$$

or writing explicitly the usual grad and div operators, it is given by

$$-H^{-1} \text{grad}(H^{-1} : \text{grad } \tilde{\mathbf{u}}) - k^2 \tilde{\mathbf{u}} = \mathbf{0}, \quad (2.2.17)$$

where the double dot notation refers to the inner product of two second-order tensors. However, Equation (2.2.17) is not adequate to obtain a simple variational formulation (due to the presence of H^{-1} multiplying the first term in (2.2.17)). Equivalently, we will multiply (2.2.17) by H , what leads to

$$-\text{grad}(H^{-1} : \text{grad } \tilde{\mathbf{u}}) - k^2 H \tilde{\mathbf{u}} = \mathbf{0}. \quad (2.2.18)$$

Remark 2.2.1. *Other equivalent partial differential equations to (2.2.17) and (2.2.18) could be also used with similar numerical discretizations. For instance, in [17], Eq. (2.2.17) is multiplied by $\det J = H$. Then, taking into account that $H = H^T$, and the fact the j -th diagonal coefficient of JH^{-T} is independent of the coordinate x_j , we have*

$$-\text{div}((JH^{-T} \otimes H^{-1}) \text{grad } \tilde{\mathbf{u}}) - k^2 J \tilde{\mathbf{u}} = \mathbf{0},$$

where let us recall that for any arbitrary second-order tensors A, B, C , the tensor product is defined by $(A \otimes B)C = A(B : C)$. Despite the PML model described above leads to a symmetric variational formulation, the present work will use (2.2.18) due to its simplicity.

One of the main elements in the PML technique relies on an adequate choice of the PML absorbing profile σ . A wide variety of research publications have been published in the last decade focused on the optimal selection of this PML profile σ . In particular, in [14], it has shown that the Cartesian PML technique for time-harmonic linear acoustic problems reaches an optimal performance if

$$\sigma_{x_j}(s) = \frac{1}{L_j^* - s}, \quad (2.2.19)$$

where $x_j = L_j^*$ is the plane where the exterior boundary of the PML layer is located. This choice of the absorbing profile ensures an optimal damping for a given ω and the continuity of the displacement field across Ω and Ω_{PML} . An analogous result is also proved in [15] for cylindrical PML layers by using the optimal radial absorbing profile

$$\sigma_r(s) = \frac{1}{R^* - s}, \quad (2.2.20)$$

where $r = R^*$ is the sphere where the exterior boundary of the PML layer is placed.

2.3 Truncation of the scattering problem

Thanks to the use of PML layers, the original acoustic scattering problem (2.2.7)-(2.2.10) stated in an unbounded domain Ω will be truncated. In that manner the new computational

domain is split in two disjoint parts: the physical region of interest Ω_F and the surrounding PML region, placed in the subdomain Ω_{PML} . The enclosures of each subdomain share a common interface boundary Γ_I where coupling conditions between displacement fields $\mathbf{u}$ and $\tilde{\mathbf{u}}$. These conditions ensure the continuity of the normal displacements (kinematic condition) and the continuity of the surface loads (kinetic conditions) between the fluid and the PML subdomains. Finally, since the outgoing solution should be damped completely by the PML layers, a null condition on all the components of displacement is imposed in the exterior PML boundary $\Gamma_\star$.

In conclusion, taking into account the fluid and PML governing equations and the boundary conditions, the acoustic scattering problem (2.2.7)-(2.2.10) truncated with the PML layers reads as follows: For a given wavenumber k , and boundary data u_D and p_N , find $(\mathbf{u}, \tilde{\mathbf{u}})$ such that

$$-\text{grad div } \mathbf{u} - k^2 \mathbf{u} = \mathbf{0} \quad \text{in } \Omega_F, \quad (2.3.1)$$

$$-\text{grad}(H^{-1} : \text{grad } \tilde{\mathbf{u}}) - k^2 H \tilde{\mathbf{u}} = \mathbf{0} \quad \text{in } \Omega_{\text{PML}}, \quad (2.3.2)$$

$$\mathbf{u} \cdot \mathbf{n} = u_D \quad \text{on } \Gamma_D, \quad (2.3.3)$$

$$-\rho_0 c_0^2 \text{div } \mathbf{u} = p_N \quad \text{on } \Gamma_N, \quad (2.3.4)$$

$$\mathbf{u} \cdot \mathbf{n} = \tilde{\mathbf{u}} \cdot \mathbf{n} \quad \text{on } \Gamma_I, \quad (2.3.5)$$

$$\text{div } \mathbf{u} = H^{-1} : \text{grad } \tilde{\mathbf{u}} \quad \text{on } \Gamma_I, \quad (2.3.6)$$

$$\tilde{\mathbf{u}} = \mathbf{0} \quad \text{on } \Gamma_\star. \quad (2.3.7)$$

Notice that the description of this truncated scattering problem is valid for Cartesian PML models. Analogous PML models can be written in spherical coordinates, only requiring to replace the tensor H and the differential operators conveniently with expressions in (2.2.11) and (2.2.16) (see for instance [12] for other arbitrary convex system of coordinates).

2.3.1 Spherical symmetric problem

If the geometrical description of the subdomains Ω_F and Ω_{PML} , and also the boundary conditions admits a radial description independently of the spherical angle coordinates θ and ϕ , the three-dimensional truncated scattering problem (2.3.1)-(2.3.7) can be reduced to a coupled one-dimensional boundary-value differential problem written in terms of the radial coordinate r , where the unique unknown field u_r is the radial component of the displacement field, i.e., $\mathbf{u} = u_r \mathbf{e}_r$. In this case, we will assume that the fluid and PML subdomains are described in spherical coordinates by

$$\Omega_F = \{(r, \theta, \phi) \in \mathbb{R}^+ \times [-\pi, \pi) \times [0, \pi) : R_D < r < R\},$$

$$\Omega_{\text{PML}} = \{(r, \theta, \phi) \in \mathbb{R}^+ \times [-\pi, \pi) \times [0, \pi) : R_D < r < R^\star\},$$

the interface boundary Γ_I is placed at $r = R$ and the exterior boundary of the PML region is situated at $r = R^\star$. Hence, the coupled problem (2.3.1)-(2.3.7) is rewritten as follows:

For a given wavenumber k , and a scalar boundary data $u_D \in \mathbb{C}$, find $(\mathbf{u}, \tilde{\mathbf{u}})$ such that

$$-\frac{d}{dr} \left(\frac{1}{r^2} \frac{d}{dr} (r^2 u_r) \right) - k^2 u_r = 0 \quad \text{in } r \in (R_D, R), \quad (2.3.8)$$

$$-\frac{d}{dr} \left(\frac{1}{\tilde{r}^2 \gamma_r} \frac{d}{dr} (\tilde{r}^2 \tilde{u}_r) \right) - k^2 \gamma_r \tilde{u}_r = 0 \quad \text{in } r \in (R, R^*), \quad (2.3.9)$$

$$u_r = u_D \quad \text{at } r = R_D, \quad (2.3.10)$$

$$u_r = \tilde{u}_r \quad \text{at } r = R, \quad (2.3.11)$$

$$\frac{du_r}{dr} = \frac{1}{\gamma_r} \frac{d\tilde{u}_r}{dr} \quad \text{at } r = R, \quad (2.3.12)$$

$$\tilde{u}_r = 0 \quad \text{at } r = R^*, \quad (2.3.13)$$

where $\gamma_r = d\tilde{r}/dr$. Notice also that from the coupling conditions (2.3.11)-(2.3.12) and due to $\tilde{r}(R) = R$, it is straightforward to check at $r = R$

$$\operatorname{div} \mathbf{u} = \frac{2}{R} u_r(R) + \frac{du_r}{dr}(R) = \frac{2}{\tilde{r}(R)} \tilde{u}_r(R) + \frac{1}{\gamma_r(R)} \frac{d\tilde{u}_r}{dr}(R) = H^{-1} : \operatorname{grad} \tilde{\mathbf{u}},$$

for $\mathbf{u} = \tilde{u}_r \mathbf{e}_r$. Hence, the coupling conditions (2.3.5)-(2.3.6) are equivalent to (2.3.11)-(2.3.12) in this spherical test. The rest of the equations in the spherical problem (2.3.8)-(2.3.13) is a direct rewriting of the original problem (2.3.1)-(2.3.7) in spherical coordinates keeping only the radial variations.

2.4 Variational formulation

Once the original unbounded domain of the scattering problem has been truncated, the coupled PML-fluid problem becomes tractable to be solved in a weak form using its variational formulation. However, the derivation of the weak problem associated with the system of equations (2.3.1)-(2.3.7) requires the use of different functional spaces for the unknown $\mathbf{u}$ and $\tilde{\mathbf{u}}$. If we read (2.3.1) from a weak point of view, it is necessary that $\mathbf{u} \in [\mathbf{L}^2(\Omega_F)]^3$ (see the second term) but also $\operatorname{div} \mathbf{u} \in \mathbf{L}^2(\Omega_F)$ (see the first term). In conclusion,

$$\mathbf{u} \in \mathbf{H}(\operatorname{div}, \Omega_F) = \{ \mathbf{w} \in [\mathbf{L}^2(\Omega_F)]^3 : \operatorname{div} \mathbf{w} \in \mathbf{L}^2(\Omega_F) \}.$$

In a similar manner, if we focus our attention on (2.3.2) from a weak point of view, $H\tilde{\mathbf{u}} \in [\mathbf{L}^2(\Omega_{\text{PML}})]^3$ and $H^{-1} : \operatorname{grad} \tilde{\mathbf{u}} \in \mathbf{L}^2(\Omega_{\text{PML}})$. Hence,

$$\tilde{\mathbf{u}} \in \mathbf{H}(\widetilde{\operatorname{div}}, \Omega_{\text{PML}}) = \{ \tilde{\mathbf{w}} \in [\mathbf{L}^2(\Omega_{\text{PML}})]^3 : H\tilde{\mathbf{w}} \in [\mathbf{L}^2(\Omega_{\text{PML}})]^3, H^{-1} : \operatorname{grad} \tilde{\mathbf{w}} \in \mathbf{L}^2(\Omega_F) \}.$$

In what follows, for the sake of simplicity in the exposition, it will be assumed that no boundary conditions are imposed in terms of the pressure p_N , and hence Γ_N will not be

considered. In this case, a classical Sobolev results (see [28] for further details) guarantees that the normal trace of displacement field in $H(\operatorname{div}, \Omega_F)$ belongs to $H^{\frac{1}{2}}(\Gamma_D)$.

Consequently, we seek the weak solution of the coupled PML-fluid scattering problem (2.3.1)-(2.3.7) in the functional space

$$V = \{(\mathbf{w}, \tilde{\mathbf{w}}) \in H(\operatorname{div}, \Omega_F) \times H(\operatorname{div}, \Omega_{\text{PML}}) : \mathbf{w} \cdot \mathbf{n} = \tilde{\mathbf{w}} \cdot \mathbf{n} \text{ on } \Gamma_I, \tilde{\mathbf{w}} = \mathbf{0} \text{ on } \Gamma_\star\}.$$

The variational formulation can be derived using standard arguments. Once the motion equations (2.3.1) and (2.3.2) are multiplied by two different test functions $(\mathbf{v}, \tilde{\mathbf{v}}) \in V$, a Green's formula is used in both equations and the boundary terms are vanishing due to the interface coupling and boundary conditions. Finally, we have the following weak problem: For a given wavenumber k , and boundary data $u_D \in H^{\frac{1}{2}}(\Gamma_D)$, find $(\mathbf{u}, \tilde{\mathbf{u}}) \in V$ such that $\mathbf{u} \cdot \mathbf{n} = u_D$ on Γ_D , and

$$\begin{aligned} \int_{\Omega_F} \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} dV - k^2 \int_{\Omega_F} \mathbf{u} \cdot \mathbf{v} dV \\ + \int_{\Omega_{\text{PML}}} (H^{-1} : \operatorname{grad} \tilde{\mathbf{u}}) \operatorname{div} \tilde{\mathbf{v}} dV - k^2 \int_{\Omega_{\text{PML}}} H \tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}} dV = 0, \end{aligned} \quad (2.4.1)$$

for all $(\mathbf{v}, \tilde{\mathbf{v}}) \in V$ such that $\mathbf{v} \cdot \mathbf{n} = 0$ on Γ_D .

2.4.1 Spherical symmetric variational problem

An analogous but much more simple variational formulation can be obtained for the scattering problem with spherical symmetry. In this case, the radial solution $(u_r, \tilde{u}_r)$ is sought in

$$\begin{aligned} V_r = \left\{ (w_r, \tilde{w}_r) \in L^2(R_D, R) \times L^2(R, R^\star) : \frac{dw_r}{dr} \in L^2(R_D, R), \right. \\ \left. \gamma_r \tilde{w}_r \in L^2(R, R^\star), \frac{1}{\gamma_r} \frac{d\tilde{w}_r}{dr} \in L^2(R, R^\star), w_r(R) = \tilde{w}_r(R), \tilde{w}_r(R^\star) = 0 \right\}. \end{aligned}$$

Now, using the system of equations (2.3.8)-(2.3.13), and taking into account the integral change of variable by using spherical spatial coordinates, the weak problem can be described as follows: For a given wavenumber k , and a scalar boundary data $u_D \in \mathbb{C}$, find $(u_r, \tilde{u}_r) \in V_r$ such that $u_r(R_D) = u_D$ and

$$\begin{aligned} \int_{R_D}^R \left(\frac{1}{r^2} \frac{d}{dr} (r^2 u_r) \right) \left(\frac{1}{r^2} \frac{d}{dr} (r^2 v_r) \right) r^2 dr - k^2 \int_{R_D}^R u_r v_r r^2 dr \\ + \int_R^{R^\star} \left(\frac{1}{\tilde{r}^2 \gamma_r} \frac{d}{d\tilde{r}} (\tilde{r}^2 \tilde{u}_r) \right) \left(\frac{1}{r^2} \frac{d}{dr} (r^2 \tilde{v}_r) \right) r^2 dr - k^2 \int_R^{R^\star} \gamma_r \tilde{u}_r \tilde{v}_r r^2 dr = 0, \end{aligned} \quad (2.4.2)$$

for all $(v_r, \tilde{v}_r) \in V_r$ such that $v_r(R_D) = 0$. Notice that implicitly, it has been used that H^1 -type functions defined on the real line are continuous (see [28]), and hence their point-wise evaluations at $r = R_D$, R , and $R^\star$ are well-defined.

2.5 Post-processing projections techniques

Since the acoustic scattering problem (2.3.1)-(2.3.7) is stated in terms of the displacement field, if we want to know the acoustic pressure field then it has to be computed *a posteriori* in a post-processing step from the compressibility condition of the acoustic fluid (2.2.2). Since this equation, which relates the pressure with the displacement field involves the divergence differential operator, the post-processing pressure computations should also be performed by using a weak formulation. In what follows, three different projection strategies are proposed.

More precisely, the pressure field can be computed using different orthogonal projection of the divergence of the displacement field into a suitable functional space using the method of variations. The specification of inner product to be used however, remains a choice and this also determines the function space that p belongs to. All of projections procedures will be focused on the pressure computation in the physical domain of interest of the scattering problem, this is, in Ω_F .

2.5.1 L^2 -projection

Taking into account that $\mathbf{u} \in H(\text{div}, \Omega_F)$, then it holds $p \in L^2(\Omega_F)$ from (2.2.2). Hence, the pressure field can be computed using the $L^2(\Omega_F)$ orthogonal projection: Find $p \in L^2(\Omega_F)$ such that

$$\int_{\Omega_F} pq \, dV = - \int_{\Omega_F} \rho_0 c_0^2 \text{div } \mathbf{u} \, q \, dV, \quad (2.5.1)$$

for all $q \in L^2(\Omega_F)$.

Despite using a L^2 -projection of Eq. (2.2.2), we can use the equation of motion (2.2.1) to show a smoother behaviour on p . Having in mind that $\mathbf{u} \in H(\text{div}, \Omega_F)$, then $\mathbf{u} \in [L^2(\Omega_F)]^3$ and Eq. (2.2.1) yields to $\text{grad } p \in [L^2(\Omega_F)]^3$, what is equivalent to $p \in H^1(\Omega_F)$. So, using this extra regularity on the pressure field, we can use a Gauss' divergence integral formula to get a similar variational problem to (2.5.1): Find $p \in H^1(\Omega_F)$ such that

$$\int_{\Omega_F} pq \, dV = \int_{\Omega_F} \rho_0 c_0^2 \mathbf{u} \cdot \text{grad } q \, dV - \int_{\partial\Omega_F} \rho_0 c_0^2 \mathbf{u} \cdot \mathbf{n} q \, dS. \quad (2.5.2)$$

Taking into account the assumption of $p \in H^1(\Omega_F)$, both L^2 -projections are equivalent. However, from a computational point of view, the solution of the projection problem (2.5.2) requires also the computation of the normal traces of the field displacement $\mathbf{u}$, which are not involved in the original scattering problem (2.3.1)-(2.3.7).

2.5.2 H^1 -projection

Following with the regularity assumptions used in the section above, a natural alternative is to consider the H^1 -projection for computing the acoustic pressure field. So, in

this case we replace the L^2 -inner product by the inner product associated with the functional space $H^1(\Omega_F)$ applied to Eq. (2.2.2) and the motion equation (2.2.4). More precisely, using (2.2.2), the motion equation (2.2.4) in the fluid domain can be rewritten as

$$\text{grad } p - \rho_0 \omega^2 \mathbf{u} = \mathbf{0} \quad \text{in } \Omega_F. \quad (2.5.3)$$

Consequently, the H^1 -projection of Eq. (2.2.2) is described as the following weak problem: Find $p \in H^1(\Omega_F)$ such that

$$\int_{\Omega_F} (pq + \text{grad } p \cdot \text{grad } q) dV = \int_{\Omega_F} ((-\rho c^2 \text{div } \mathbf{u})q + \rho_0 \omega^2 \mathbf{u} \cdot \text{grad } q) dV, \quad (2.5.4)$$

for all $q \in H^1(\Omega_F)$. Notice that the computation of this H^1 -projection involves in its right-hand side physical quantities such as $\mathbf{u}$ and $\text{div } \mathbf{u}$, which are either directly accessible from the scattering problem (2.3.1)-(2.3.7) or they are already used in the L^2 -projection.

Alternatively, since we are assuming that $p \in H^1(\Omega_F)$, it is possible to derive an equivalent version of (2.5.4) using the governing equation of the compressible fluid written in terms of the acoustic pressure. From Eqs. (2.2.1) and (2.2.2), it is immediate to check that

$$-\Delta p - k^2 p = 0 \quad \text{in } \Omega_F.$$

Hence, using standard arguments is also straightforward to write a weak form associated with the previous equation, this is, it holds

$$\int_{\Omega_F} \text{grad } p \cdot \text{grad } q dV = \int_{\Omega_F} k^2 p q dV + \int_{\partial\Omega_F} \frac{\partial p}{\partial \mathbf{n}} q dS, \quad (2.5.5)$$

for all $q \in H^1(\Omega_F)$. Finally, computing the normal component of (2.5.3) and replacing it in the boundary term of (2.5.5), the second term in the left-hand side of (2.5.4) can be replaced by (2.5.5) and we have a simplified version of the H^1 -projection: Find $p \in H^1(\Omega_F)$ such that

$$\int_{\Omega_F} (1 + k^2) p q dV = \int_{\Omega_F} ((-\rho c^2 \text{div } \mathbf{u})q + \rho_0 \omega^2 \mathbf{u} \cdot \text{grad } q) dV - \int_{\partial\Omega_F} \rho_0 \omega^2 \mathbf{u} \cdot \mathbf{n} q dS, \quad (2.5.6)$$

for all $q \in H^1(\Omega_F)$. Despite this new projection problem is equivalent to a canonical H^1 -projection, it is not required to compute any derivative on the trial pressure field p . So, essentially (except for the constant factor $1 + k^2$), the left-hand side of (2.5.6) is equivalent to the left-hand side of the L^2 -projections (2.5.1) or (2.5.2).

2.5.3 H^m -projection with $m \geq 2$

The smoothness of the acoustic pressure field depends on the regularity of the boundary data $u_D \in H^{\frac{1}{2}}(\Gamma_D)$, which is related to the normal displacements on Γ_D . If the boundary data is smoother then it is possible to assume some extra regularity on $\mathbf{u}$ and p and

hence to extend the previous procedure of the H^1 -projection to a smoother (and hence smaller) Sobolev space. More precisely, let us assume that $p \in H^m(\Omega_F)$, or equivalently $\operatorname{div} \mathbf{u} \in H^m(\Omega_F)$ with $m \geq 2$. In this case, we can apply the divergence operator on the fluid motion equation (2.2.4) to obtain

$$\Delta p = \operatorname{div} \operatorname{grad} p = \rho_0 \omega^2 \operatorname{div} \mathbf{u} = k^2 p,$$

where it has been used (2.2.2). If we consider the relation above to compute successively the spatial derivatives of the pressure field, we have

$$\Delta^j p = k^2 \Delta^{j-1} p = k^{2j} p = k^{2j} (-\rho_0 c_0^2 \operatorname{div} \mathbf{u}), \quad (2.5.7)$$

$$\operatorname{grad}(\Delta^j p) = \operatorname{grad} k^{2j} p = k^{2j} \rho_0 \omega^2 \mathbf{u}. \quad (2.5.8)$$

for all $j \in \mathbb{N}$ such that $2j + 1 \leq m$. We will consider that the $H^m(\Omega_F)$ functional space is endowed with the following inner product: if m is odd, the inner product is defined by

$$\langle p, q \rangle_{H^m(\Omega_F)} = \int_{\Omega_F} \left(pq + \sum_{j=1}^{\frac{m-1}{2}} (\operatorname{grad}(\Delta^{j-1} p) \cdot \operatorname{grad}(\Delta^{j-1} q) + \Delta^{j-1} p \Delta^{j-1} q) \right) dV, \quad (2.5.9)$$

and, if m is even, it is given by

$$\langle p, q \rangle_{H^m(\Omega_F)} = \int_{\Omega_F} \left(pq + \sum_{j=1}^{\frac{m}{2}} (\operatorname{grad}(\Delta^{j-1} p) \cdot \operatorname{grad}(\Delta^{j-1} q) + \Delta^j p \Delta^j q) \right) dV. \quad (2.5.10)$$

Notice that in the expression written above Δ^0 denotes the identity mapping.

Now, taking into account the relations (2.5.7) and (2.5.8) in the definition of the H^m -inner products described above yields to the following two projections problems (depending on the even or odd value of m): if m is odd, find $p \in H^m(\Omega_F)$ such that

$$\begin{aligned} \langle p, q \rangle_{H^m(\Omega_F)} &= \int_{\Omega_F} (-\rho_0 c_0^2 \operatorname{div} \mathbf{u}) q dV \\ &+ \int_{\Omega_F} \sum_{j=1}^{\frac{m-1}{2}} (k^{2(j-1)} \rho_0 \omega^2 \mathbf{u} \cdot \operatorname{grad}(\Delta^{j-1} q) + k^{2(j-1)} (-\rho_0 c_0^2 \operatorname{div} \mathbf{u}) \Delta^j q) dV, \end{aligned} \quad (2.5.11)$$

for all $q \in H^m(\Omega_F)$. Otherwise, if m is even, find $p \in H^m(\Omega_F)$ such that

$$\begin{aligned} \langle p, q \rangle_{H^m(\Omega_F)} &= \int_{\Omega_F} (-\rho_0 c_0^2 \operatorname{div} \mathbf{u}) q \\ &+ \int_{\Omega_F} \sum_{j=1}^{\frac{m}{2}} (k^{2j} \rho_0 \omega^2 \mathbf{u} \cdot \operatorname{grad}(\Delta^{j-1} q) + k^{2j} (-\rho_0 c_0^2 \operatorname{div} \mathbf{u}) \Delta^j q) dV, \end{aligned} \quad (2.5.12)$$

for all $q \in H^m(\Omega_F)$. Notice again that the computation of this H^m -projection with $m \geq 2$ involves in its right-hand side physical quantities such as $\mathbf{u}$ and $\operatorname{div} \mathbf{u}$, which are either direct accessible from the scattering problem (2.3.1)-(2.3.7) or they are already used in the L^2 -projection. So, despite a high-order inner product is used (requiring m -order spatial derivatives for the trial pressure field p in the left-hand side), the displacement field $\mathbf{u}$ does not require the computation of those high-order derivatives.

Using analogous arguments to those ones applied for the H^1 -projection, it could be possible to simplify the left-hand side of (2.5.11) and (2.5.12) such that it could involve only an L^2 -inner product of pressure fields. However, this procedure will have the drawback of introducing around $m/2$ new boundary integrals written in terms of the normal displacements on $\partial\Omega_F$.

2.6 Discretization

Since the scattering problem (2.3.1)-(2.3.7) and subsequently, its variational formulation is written in terms of the displacement field as primal unknown, a suitable discretization by using a Finite Element Method (FEM) should be proposed also in terms of the displacement field. It is well-known that a standard finite element discretization based on piecewise linear elements at each vector component leads to some spurious errors, due to the presence of rotational modes in the kernel of the divergence term in (2.4.1) (see [22] for a detailed discussion). In addition, the use of that discretization leads to a huge number of degrees of freedom in comparison with a standard piecewise linear discretization of the same scattering problem written in terms of the pressure field. To avoid those discretization drawbacks, Raviart-Thomas Finite Elements must be used for displacement-based formulations (see [22, 34]).

In the present work, first-order Raviart-Thomas elements will be considered in a simplex mesh $\mathcal{T}_h$, satisfying $\Omega = \cup_{T \in \mathcal{T}_h} T$, and such that it is conformal with the boundary partition and the interface boundary Γ_I between the fluid and PML subdomains. Hence, the discrete space V_h is given by

$$\begin{aligned} V_h = \{(\mathbf{w}_h, \tilde{\mathbf{w}}_h) \in V : \\ \mathbf{w}|_T = \mathbf{a}_T + b_T(x_1, x_2, x_3)^T, \mathbf{a}_T \in \mathbb{C}^3, b_T \in \mathbb{C}, T \subset \Omega_F, T \in \mathcal{T}_h \\ \tilde{\mathbf{w}}|_T = \tilde{\mathbf{a}}_T + \tilde{b}_T(x_1, x_2, x_3)^T, \tilde{\mathbf{a}}_T \in \mathbb{C}^3, \tilde{b}_T \in \mathbb{C}, T \subset \Omega_{\text{PML}}, T \in \mathcal{T}_h\} \quad (2.6.1) \end{aligned}$$

Additionally, the boundary data of the scattering problem $u_D \in H^{\frac{1}{2}}(\Gamma_D)$ should be discretized. To be coherent with the Raviart-Thomas discretization, since $\mathbf{w}_h \cdot \mathbf{n}$ is constant on the boundary of the elements of the simplicial mesh for any $\mathbf{w}_h \in V_h$, then u_D is approximated by its piecewise constant projection $u_{h,D}$ on the faces of the mesh lying on Γ_D .

Having in mind the definition of V_h written above, the discrete problem associated with the scattering problem (2.3.1)-(2.3.7) is described as follows: For a given wavenumber k , and

discrete piecewise constant boundary data $u_{h,D}$, find $(\mathbf{u}_h, \tilde{\mathbf{u}}_h) \in V_h$ such that $\mathbf{u}_h \cdot \mathbf{n} = u_{h,D}$ on Γ_D , and

$$\begin{aligned} \int_{\Omega_F} \operatorname{div} \mathbf{u}_h \operatorname{div} \mathbf{v}_h dV - k^2 \int_{\Omega_F} \mathbf{u}_h \cdot \mathbf{v}_h dV \\ + \int_{\Omega_{\text{PML}}} (H^{-1} : \operatorname{grad} \tilde{\mathbf{u}}_h) \operatorname{div} \tilde{\mathbf{v}}_h dV - k^2 \int_{\Omega_{\text{PML}}} H \tilde{\mathbf{u}}_h \cdot \tilde{\mathbf{v}}_h dV = 0, \end{aligned} \quad (2.6.2)$$

for all $(\mathbf{v}_h, \tilde{\mathbf{v}}_h) \in V_h$ such that $\mathbf{v}_h \cdot \mathbf{n} = 0$ on Γ_D .

To write the matrix description of the discrete variational problem (2.6.2), we need to introduce a basis of V_h . Due to the definition of the Raviart-Thomas discretization (where the normal component of the discrete functions are constant on the faces of the simplicial mesh), let us define the basis of V_h , denoted by $\{\boldsymbol{\varphi}_j\}_{j=1}^n$, such that

$$\boldsymbol{\varphi}_j \cdot \mathbf{n}|_{F_i} = \delta_{ij} \quad 1 \leq i, j \leq n,$$

where n is the total number of faces in the mesh $\mathcal{T}_h$, F_i is the i -th face of the simplicial mesh, and δ_{ij} is the Kronecker's delta. In this manner, any function pair can be represented in terms of this basis as $\mathbf{w}_h = \sum_{j=1}^{n_F} w_j \boldsymbol{\varphi}_j$ and $\tilde{\mathbf{w}}_h = \sum_{j=1}^{n_{\text{PML}}} \tilde{w}_j \boldsymbol{\varphi}_{j+n_F}$, where the degrees of freedom associated with the discretization has been split in two disjoint index sets: $\{1, \dots, n_F\}$ corresponds to the faces of the mesh in the fluid subdomain Ω_F and $\{n_F, \dots, n_F + n_{\text{PML}}\}$ denotes the indices of the faces in the PML subdomain. Notice that $n_F + n_{\text{PML}} > n$ because the mesh in Ω_F and Ω_{PML} share a finite number of faces in the coupling interface Γ_I . The vectors containing the linear coefficients w_j and $\tilde{w}_j$ of $\mathbf{w}_h$ and $\tilde{\mathbf{w}}_h$ are denoted by $\vec{w}_h$ and $\vec{\tilde{w}}_h$, respectively.

Finally, the discrete variational problem (2.6.2) can be written in matrix form as follows:

$$\left(\begin{pmatrix} \mathcal{K}_F & \mathbf{0} \\ \mathbf{0} & \mathcal{K}_{\text{PML}}(k) \end{pmatrix} - k^2 \begin{pmatrix} \mathcal{M}_F & \mathbf{0} \\ \mathbf{0} & \mathcal{M}_{\text{PML}}(k) \end{pmatrix} \right) \begin{pmatrix} \vec{w}_h \\ \vec{\tilde{w}}_h \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix}, \quad (2.6.3)$$

where the block matrices of the discrete system described above are defined by

$$\begin{aligned} [\mathcal{K}_F]_{ij} &= \int_{\Omega_F} \operatorname{div} \boldsymbol{\varphi}_i \operatorname{div} \boldsymbol{\varphi}_j dV & 1 \leq i, j \leq n_F, \\ [\mathcal{K}_{\text{PML}}]_{ij} &= \int_{\Omega_{\text{PML}}} (H^{-1} : \operatorname{grad} \boldsymbol{\varphi}_{i+n_F}) \operatorname{div} \boldsymbol{\varphi}_{j+n_F} dV & 1 \leq i, j \leq n_{\text{PML}}, \\ [\mathcal{M}_F]_{ij} &= \int_{\Omega_F} \boldsymbol{\varphi}_i \cdot \boldsymbol{\varphi}_j dV & 1 \leq i, j \leq n_F, \\ [\mathcal{M}_{\text{PML}}]_{ij} &= \int_{\Omega_{\text{PML}}} H \boldsymbol{\varphi}_{i+n_F} \cdot \boldsymbol{\varphi}_{j+n_F} dV & 1 \leq i, j \leq n_{\text{PML}}. \end{aligned}$$

Let us remark that the linear matrix system (2.6.3) involves a $n \times n$ matrices, since part of the block matrices and the unknown vectors $\vec{w}_h$ and $\vec{\tilde{w}}_h$ are assembled at the same position (the shared degrees of freedom located on the coupling interface Γ_I).

In conclusion, the scattering problem (2.3.1)-(2.3.7) can be solved using a finite element approximation, which can discretize the Sobolev function space based on a spatial simplicial mesh and reduce the weak form (2.4.1) into its matricial form. This provides the means of arriving at the approximation of displacement field solution for the Helmholtz problem using a FEM discretization.

2.6.1 Discrete post-processing techniques

Obviously the post-processing projection procedures, which have been described in Section 2.5, have to be also discretized to compute the pressure field by means of a Finite Element Method. Depending on the chosen projection, the functional space used as framework for the pressure field is different and hence, also the discrete space in the finite element approximation will be different.

In the case of the L^2 -projection (2.5.1), since $p \in L^2(\Omega_F)$, the natural choice for this approximation is a piecewise discontinuous constant space, this is, the FEM pressure approximation is sought in the discrete space

$$Q_{0h} = \{q_h \in L^2(\Omega_F) : q_h|_T = a, a \in \mathbb{C}, T \in \mathcal{T}_h\}.$$

This choice of the finite element discrete space is consistent with the Raviart-Thomas discrete space chosen for discretizing the displacement field since $\text{div } \mathbf{w}_h$ is constant per element for all $\mathbf{w}_h \in V_h$, and hence, compatible with the discretization proposed for the pressure field. The discrete L^2 -projection is described as follows: Find $p_h \in L^2(\Omega_F)$ such that

$$\int_{\Omega_F} p_h q_h dV = - \int_{\Omega_F} \rho_0 c_0^2 \text{div } \mathbf{u}_h q_h dV, \quad (2.6.4)$$

for all $q \in Q_{0h}$. A special and interesting feature of this kind of piecewise constant discretizations relies on the fact that the computation of the p_h values can be made locally, since the left and the right-hand side of (2.6.4) are constant per element.

In the case of the H^1 -projection, a typical piecewise linear finite element approximation will be used to compute the discrete pressure field. Despite this is not totally coherent with the first-order Raviart-Thomas discretization, the smoothness required due to the assumption $p \in H^1(\Omega_F)$ implies that the discrete functions must be continuous. Consequently, the pressure field will belong to

$$Q_{1h} = \{q_h \in \mathcal{C}(\Omega_F) : q_h|_T \in \mathbb{P}_1(\mathbb{C}), T \in \mathcal{T}_h\},$$

and the discrete H^1 -projection problem is described as follows: Find $p_h \in Q_{1h}$ such that

$$\begin{aligned} \int_{\Omega_F} (1 + k^2) p_h q_h dV = \int_{\Omega_F} ((-\rho c^2 \text{div } \mathbf{u}_h) q_h + \rho_0 \omega^2 \mathbf{u}_h \cdot \text{grad } q_h) dV \\ - \int_{\partial\Omega_F} \rho_0 \omega^2 \mathbf{u}_h \cdot \mathbf{n} q_h dS, \end{aligned} \quad (2.6.5)$$

for all $q_h \in Q_{1h}$.

Despite no differential operators are acting on the trial discrete function p_h , it is essential to keep a continuous discretization on p_h , since the right-hand side of (2.5.6) requires the computation of gradients of the test functions. Moreover, all the projections computations have been stated in the entire domain Ω_F . However, this characteristic is not strictly necessary, since the normal displacement field is known (computed) at any set of faces of the simplicial mesh and hence, both projections method could be applied to a size-reduced domain of interest.

Finally, let us remark also that the high-order H^m -projections with $m \geq 2$ cannot be implemented with standard continuous piecewise linear finite element discretizations. In fact, since it is assumed that $p \in H^m(\Omega_F)$ then the discrete space should be contained at least in $\mathcal{C}^1(\Omega_F)$, which is only satisfied if Hermite-type finite element approximation (see [34] for more details) are used. Consequently, since the use of this type of finite elements are far from the scope of the present work, H^m -projections with $m \geq 2$ will not be included in the numerical results of the following section.

2.7 Numerical comparison

A test case for the Helmholtz problem needs to be constructed in order to evaluate the accuracy of computing pressure from the FEM solution of the displacement field. The sound field created by an monopole source is chosen, motivated by its easy replication of the various boundary conditions encountered in different spatial dimensions. Two problems are constructed based on the fixed and open boundary conditions as are usually found in various applications. The bounded problem (labelled as “BVP test”) contains fixed ends on either ends of the one-dimensional domain. The second test is based on an unbounded problem which has a PML layer on one of the ends (labelled as “PML test”). The boundary conditions are specified using the monopole expressions and both problems are stated and solved using the finite element method to retrieve the exact solution of the monopole field. The pressure is also computed from the resulting displacement field and compared against the exact expression for sound pressure.

A monopole is an acoustic point source emitting radial waves. Consider a monopole placed at origin (outside the computational domain), and the physical domain of interest, $\Omega_F = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : R_D < r = \sqrt{x_1^2 + x_2^2 + x_3^2} < R\}$. The pressure field produced by the monopole follows the expression for the pressure field (written in spherical coordinates),

$$p_{\text{ex}}(r, \theta, \phi) = p_0 \frac{e^{-ikr}}{r}, \quad (2.7.1)$$

where p_0 is the amplitude of the monopole, and the expression for the displacement field,

$$\mathbf{u}_{\text{ex}}(r, \theta, \phi) = \frac{p_0}{\rho_0 \omega^2} \frac{e^{-ikr}}{r^2} (-1 - ikr) \mathbf{e}_r. \quad (2.7.2)$$

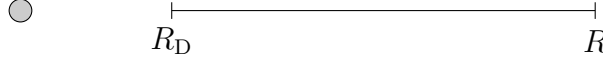


Figure 2.1: Radial sketch of a spherical symmetric unbounded problem truncated without PML layers where $R_D > 0$.

The problem truncated without PMLs is constructed by specifying the radial displacements (normal component to the spherical boundaries) at the ends of the computational domain, i.e., at $r = R_D$ and $r = R$. The Dirichlet boundary conditions u_D and u_I are obtained from the fundamental solution (2.7.2), whereas the variational discrete problem will inherit the weak formulation from the radial version of the variational problem (2.4.2) taking into account the spherical symmetry of the geometry.

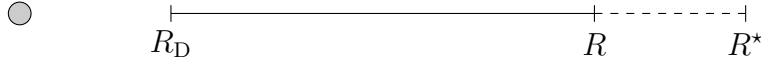


Figure 2.2: Radial sketch of a spherical symmetric unbounded problem truncated with a PML layer where $R_D > 0$.

The second test which has been used to illustrate the performance of the pressure projections techniques is focused on solving the scattering problem stated in a computational domain with spherical symmetry (2.3.11)-(2.3.12). Let us recall that in this case, a complex-valued stretching of the spatial coordinates (in this case the radial coordinate) is introduced to derive the governing equation in the PML layer.

In both test cases, once the exact solution of the monopole field is approximated by using the finite element method, the relative $L^2(\Omega_F)$ -errors can be computed for displacement field as,

$$\xi_h(\mathbf{u}_h) = 100 \frac{\|\mathbf{u}_h - \mathcal{J}_h \mathbf{u}_{\text{ex}}\|_{L^2(\Omega_F)}}{\|\mathcal{I}_h \mathbf{u}_{\text{ex}}\|_{L^2(\Omega_F)}} = \sqrt{\frac{(\vec{u}_h - \vec{u}_{\text{ex},h})^* M_F (\vec{u}_h - \vec{u}_{\text{ex},h})}{\vec{u}_{\text{ex},h}^* M_F \vec{u}_{\text{ex},h}}} \quad (\%),$$

where $\mathcal{J}_h \mathbf{u}_{\text{ex}}$ is the interpolant associated with the discrete space V_h (see [45] for further details) of the exact solution and $\vec{u}_{\text{ex},h}$ is the vector containing the values of the degrees of freedom. Similarly, the error can be computed for the pressure field by using

$$\xi_h(\mathbf{p}_h) = 100 \frac{\|p - \mathcal{I}_h p_{\text{ex}}\|_{L^2(\Omega_F)}}{\|\mathcal{I}_h p_{\text{ex}}\|_{L^2(\Omega_F)}} = \sqrt{\frac{(\vec{p}_h - \vec{p}_{\text{ex},h})^* M_F^Q (\vec{p}_h - \vec{p}_{\text{ex},h})}{\vec{p}_{\text{ex},h}^* M_F^Q \vec{p}_{\text{ex},h}}} \quad (\%),$$

where $\mathcal{I}_h \mathbf{u}_{\text{ex}}$ is the Lagrange interpolant of the exact solution associated with the mesh vertices in the case of the discrete space Q_{1h} or associated with the centre of the elements in the case of the discrete space Q_{1h} . The vectors $\vec{p}_h$ and $\vec{u}_{\text{ex},h}$ are the vectors containing the values of the degrees of freedom corresponding to the discrete approximations of the pressure fields p_h and $\mathcal{I}_h \mathbf{u}_{\text{ex}}$, respectively. Finally, the matrix M_F^Q is the finite element mass matrix associated with the left-hand side term in (2.6.4) or (2.6.5).

2.7.1 Errors induced by the PML layer over frequency

The spherical computational domain was designed with $R_D = 0.25$ and $R = 1$. The physical domain of interest was discretized with a variety of meshes with different number of vertices, this is, $N = 3 \times 2^m$, $6 \leq m \leq 10$. The two test monopole problems described above were solved numerically for frequencies between $f = 20$ Hz and $f = 1200$ Hz sampled at equispaced frequencies. At those frequencies, the errors related to the displacement and the pressure fields were computed.

The BVP test case implements the bounded problem with Dirichlet boundary conditions imposed on both ends of the domain. The PML test case is implemented with Dirichlet condition at $r = R_D$, and the perfectly matched layer of different widths with an inner (coupling) boundary placed at $r = R$.

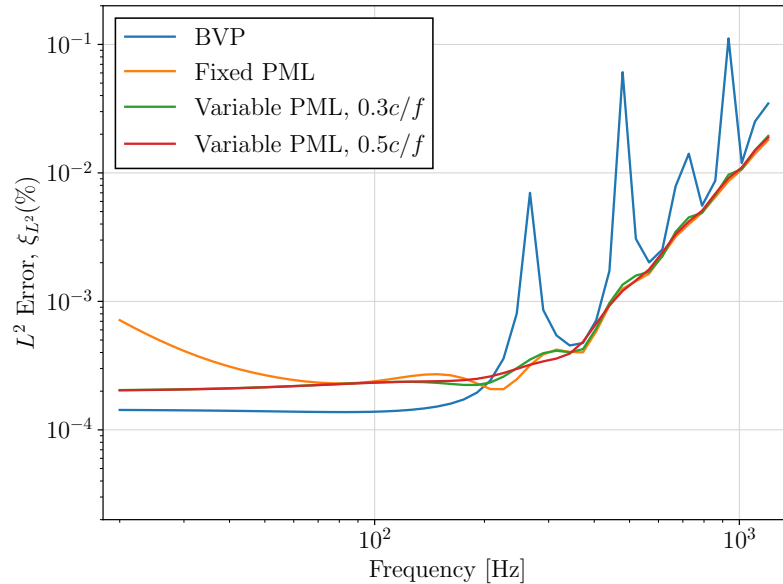


Figure 2.3: Errors in displacement field for the spherical-symmetric computational domain with $N = 3 \times 2^8$ vertices for different configurations of the BVP and PML problems.

Firstly, different PML widths are considered to evaluate the influence it bears on the accuracy of the displacement approximations. A fixed PML domain located between the spheres $r = R = 1.0$ and $r = R^* = 1.25$, is considered against frequency k -dependent PML layers, with $R^* = 1.0 + 0.3k$ and $R^* = 1.0 + 0.5k$, being $k = c_0/f$ the wavenumber. The results compared initially are the harmonic response of the error in displacement field for the various cases (see Figure 2.3). As it could be expected, the numerical results show that the displacement field obtained in the BVP test exhibits resonances at certain positive frequencies. This corresponds to the natural eigenfrequencies of the bounded spherical domain, which are not shown in the PML problems since the scattering problem is well-posed for any positive frequency.

To compute the approximations related with the pressure field, L^2 - and H^1 -projections

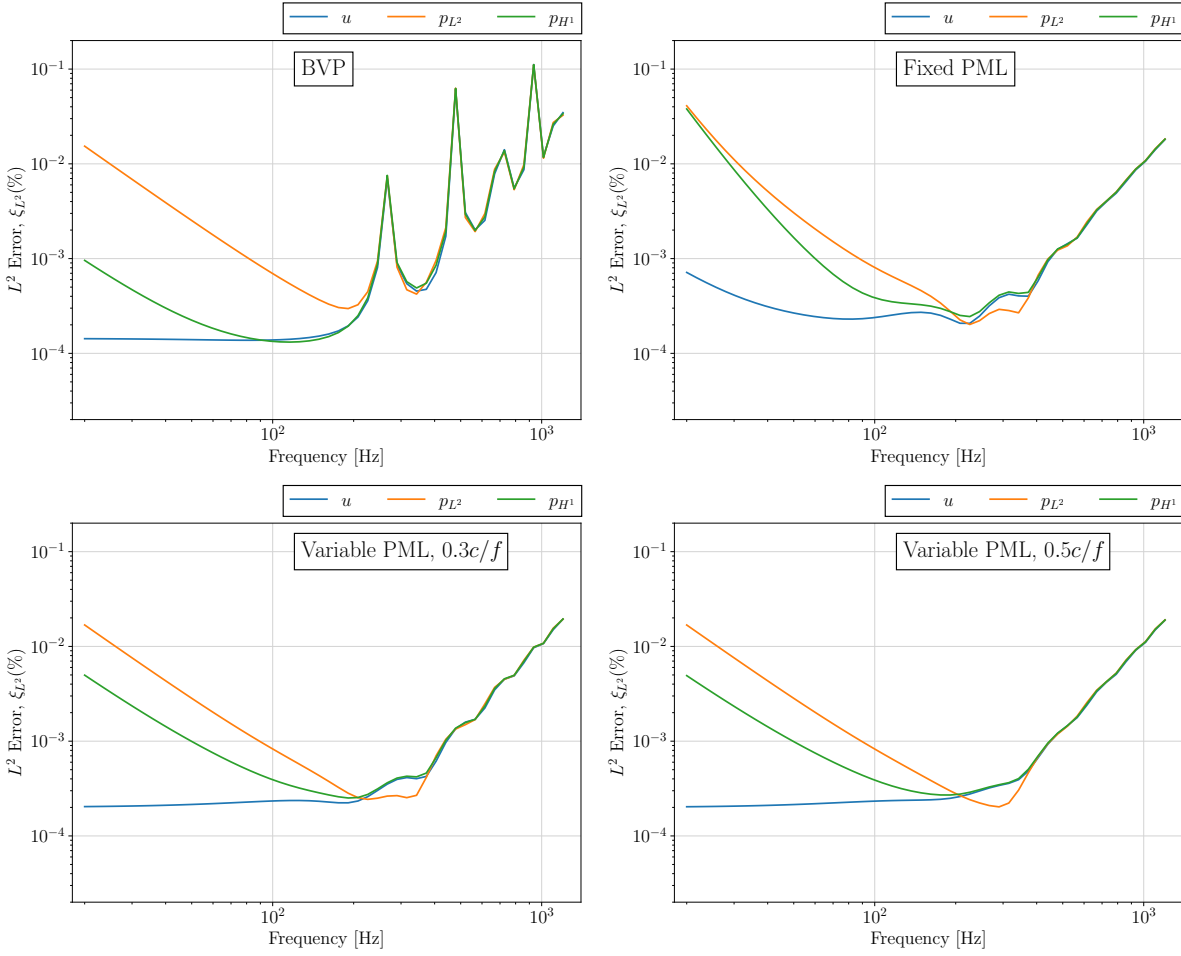


Figure 2.4: Errors in pressure field for a spherical-symmetric computational domain discretized with a mesh of $N = 3 \times 2^8$ points in the BVP test case (*top-left*), the fixed-width PML test (*top-right*) problem and for the variable PML test case with thickness $0.3c/f$ (*bottom-left*) and $0.5c/f$ (*bottom-right*).

have been used considering the finite element spaces Q_{0h} (discontinuous piecewise constant) and Q_{1h} (continuous piecewise linear), respectively. Figure 2.4 shows the comparison of errors for L^2 and H^1 projections for the BVP test case and the PML test case with a fixed thickness (independent of k). For the BVP problem, the H^1 -projection displays up to one order of magnitude more of accuracy than the L^2 -projection at lower frequencies. Similar results are observed for the variable-width PML problem, indicating that the errors at lower frequencies were dominated by the effectiveness of the PML. At higher frequencies, the L^2 and H^1 projections perform similarly for all problems since the spurious dispersion errors due to a poor finite element discretization are polluting the numerical approximation.

Notice that the pressure projections obtained for the PML test cases do not display any resonances as was observed with the displacement field. However, solving the entire

pressure Helmholtz equation with the boundary values obtained from the displacement field in a bounded domain is equivalent to solve a BVP test case formulated in terms of the pressure field and hence it would exhibit real-valued resonances.

2.7.2 Local and naively parallel projections

Since the L^2 - and H^1 -projection techniques can be used to compute the entire pressure field in Ω_F as well as in local subdomains of a reduced size contained in the physical domain of interest, we have chosen three different subdomains to test the error localization in the post-processing projection techniques. With this purpose, the following three subdomains have been selected attending to be close to $r = R_D$, in the middle of the physical domain of interest, and close to the PML layer: (a) subdomain between $r = R_D = 0.25$ and $r = 0.3$, (b) subdomain between $r = 0.5$ and $r = 0.75$, and (c) subdomain between $r = 0.9$ and $r = R = 1$. The comparison of the pressure projections for the subdomain (a) closer to the Dirichlet boundary is shown in Figure 2.5.

No relevant differences are observed in the accuracy between the two different projections. In fact, in most of the cases, the H^1 -projection reaches lower errors, however, the differences between both projections strategies are almost negligible at lower frequencies in most of the test case scenarios, as demonstrated in Figures 2.6 and 2.7.

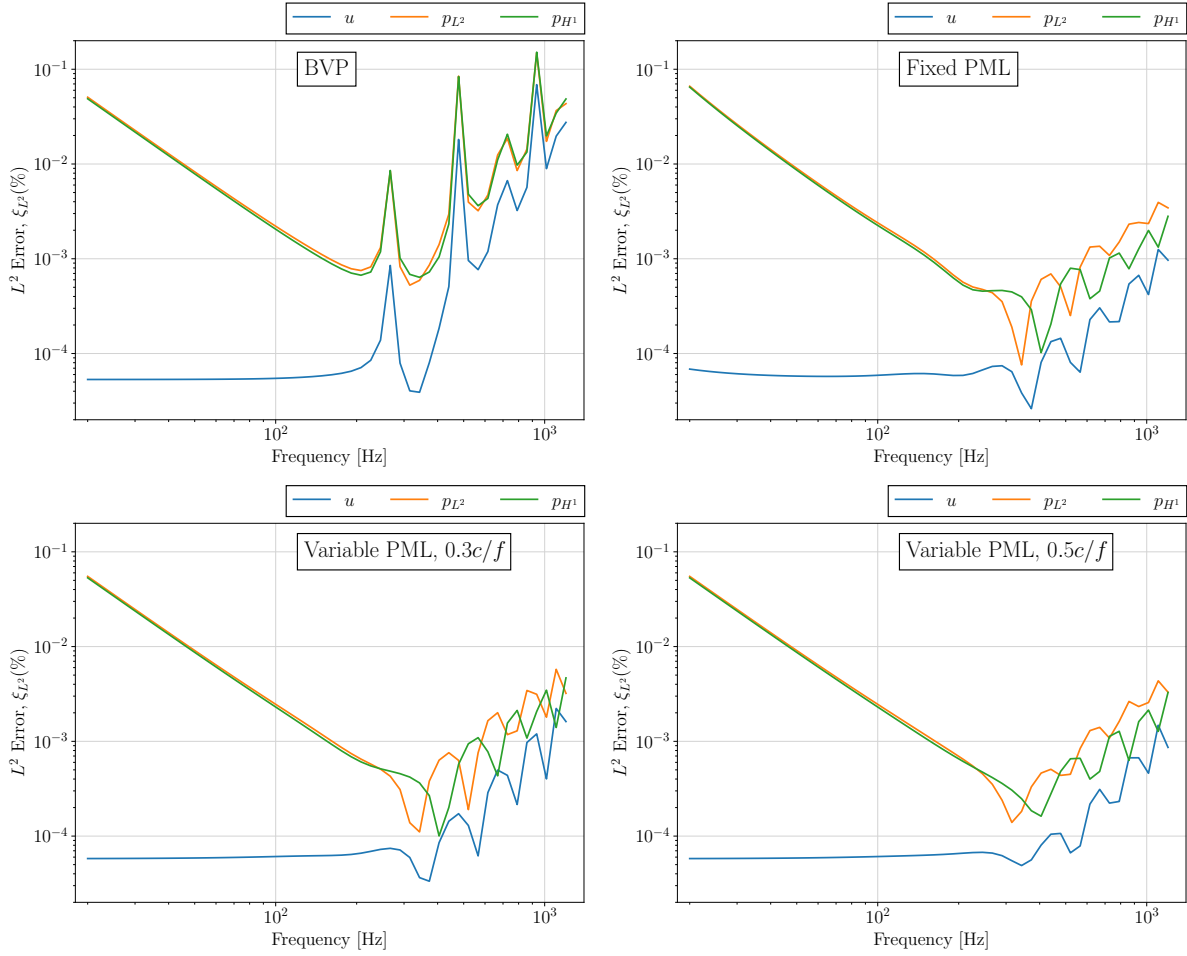


Figure 2.5: Errors in pressure field for subdomain (a) between $r = 0.25$ and $r = 0.3$, in the BVP test case (*top-left*), the fixed-width PML test (*top-right*) problem and for the variable PML test case with thickness $0.3c/f$ (*bottom-left*) and $0.5c/f$ (*bottom-right*).

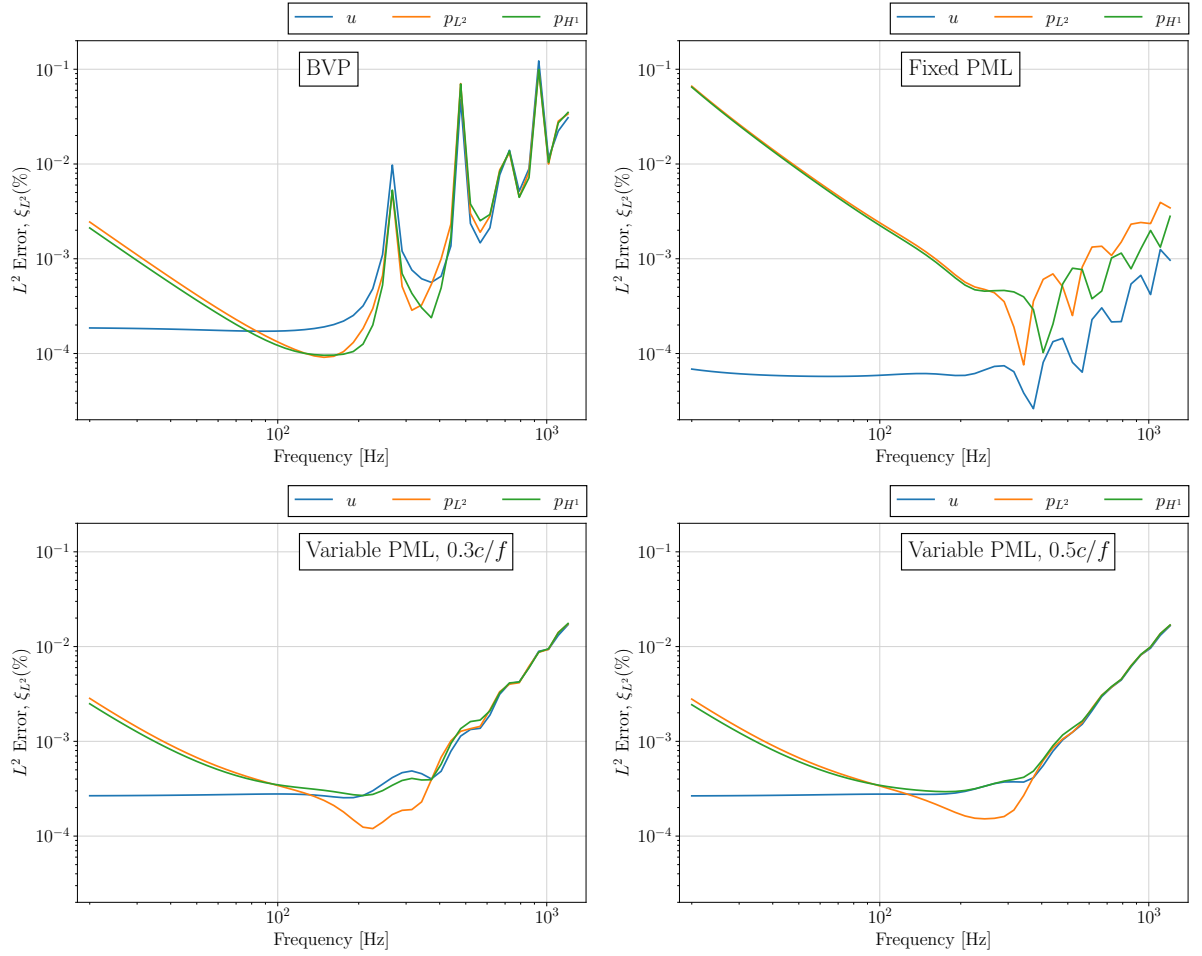


Figure 2.6: Errors in pressure field for subdomain (b) between $r = 0.5$ and $r = 0.75$, in the BVP test case (*top-left*), the fixed-width PML test (*top-right*) problem and for the variable PML test case with thickness $0.3c/f$ (*bottom-left*) and $0.5c/f$ (*bottom-right*).

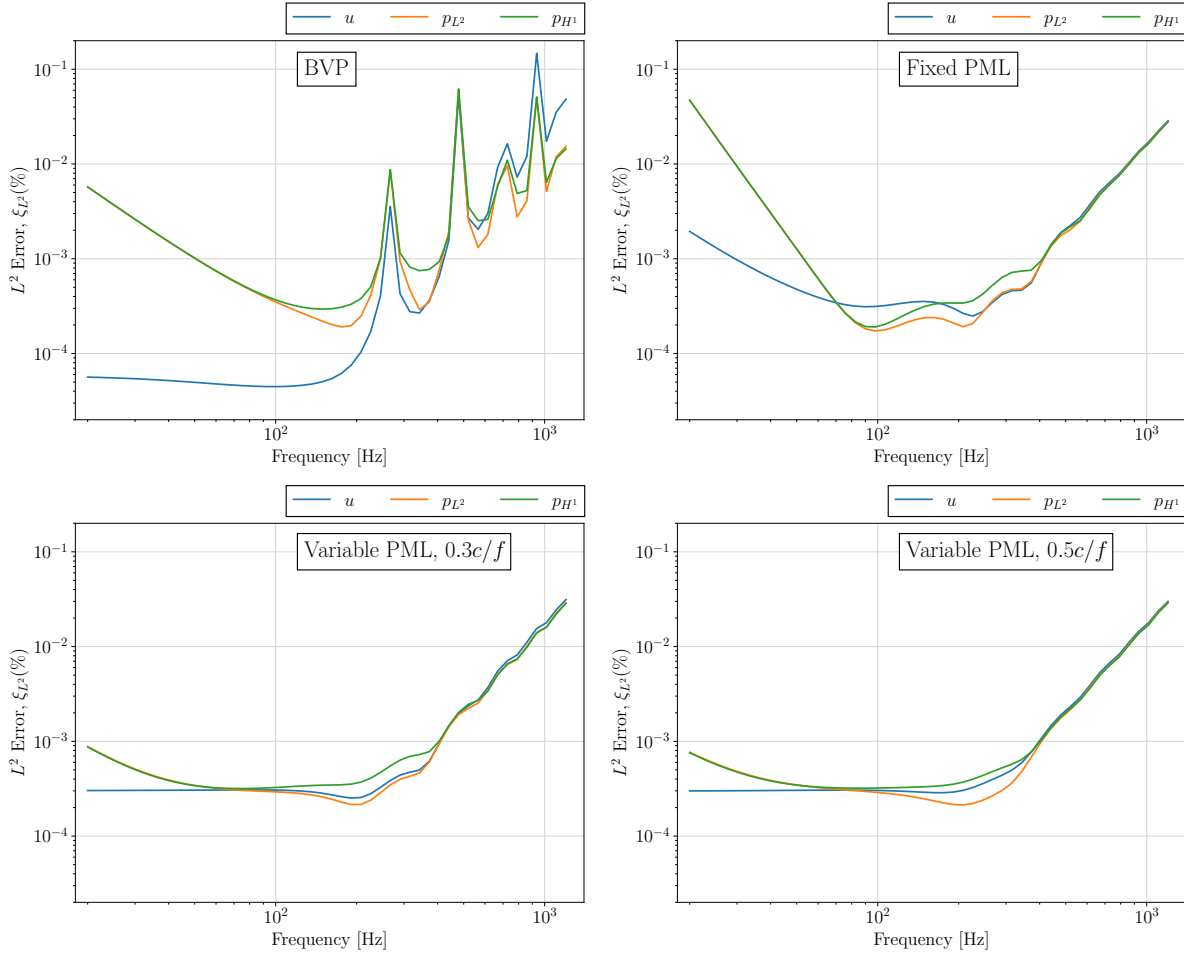


Figure 2.7: Errors in pressure field for subdomain (c) between $r = 0.9$ and $r = 1.0$, in the BVP test case (*top-left*), the fixed-width PML test (*top-right*) problem and for the variable PML test case with thickness $0.3c/f$ (*bottom-left*) and $0.5c/f$ (*bottom-right*).

2.8 Conclusions

For those mechanical or vibro-acoustic problems which are described in terms of the displacement field, not only the accurate computation of the primal unknown fields are important but also the indirectly derived physical quantities which can contain valuable information, such as the pressure field or the intensity field. In particular, the present work presents a numerical comparison of two different post-processing techniques based on different projection methods to compute the pressure field in displacement-based models. The accuracy of the proposed techniques is described in detail and the numerical results are discussed and compared in terms of two test case scenarios: an acoustic problem stated in an bounded domain and a scattering problem stated in a truncated domain with a PML layer. The H^1 -projection shows a clear advantage when the pressure field is computed in the entire physical domain of interest. However, if the pressure field is only calculated in a subdomain of the computational domain, the differences between projection techniques become negligible.

Chapter 3

Modelling acoustic insertion loss of steel meshes and windscreens for sound pressure and particle velocity sensors

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3.1 Introduction

Acoustic measurements are mostly far from straightforward, often involving unexpected difficulties regarding the sound emission source(s) and its environment. Sound sources can be physically inhibiting or in motion, preventing measurements in its proximity. They may also impose a necessity to measure in outdoor environments, in presence of wind or rain. These situations arise in practical applications such as measuring ground vehicles, wind turbines, aircrafts and require testing procedures adapted to each specific case.

For measurements done outdoors in presence of wind, additional shielding is necessary. Air flow can induce noise in the sensor measurements due to turbulence and can pose problems in quantifying the acoustic field [25, 29]. The susceptibility of an acoustic sensor to wind noise is dependent on its mechanical design and transduction principle. Pressure-gradient and particle-velocity based acoustic transducers exhibit more flow-induced noise compared to pressure-based transducers [29, 38, 42]. To attenuate this noise and improve measurement quality, the sensors are covered with windscreens mostly made of porous materials. A thorough understanding of sound transmission through windscreens is required for its design to ensure an accurate measurement.

Microphone windscreens started as silk-wrapped cages in the early patents of Thorne [69] and Spotts [67] while their early spherical designs were inspired by theoretical studies of Phelps [64] who utilized Bernoulli's principle to provide an expression for air pressure around a spherical microphone assembly. The spherical form factor was attributed to direction-neutrality towards wind noise attenuation. A subsequent report by Hosier and Donovan [48] investigated the performance of ten different microphone windscreens and offered valuable insights into windscreen design. It highlighted key descriptors in designing a windscreen: *wind-induced noise* referring to the noise produced by air flow and *acoustic insertion loss* indicating the difference in sound pressure levels measured with and without the windscreen. It also suggested the better performance of foam materials towards noise attenuation and subsequent windscreens have preferred the porous foam materials for their structural characteristics and better signal-to-noise ratio (SNR) [75].

Recent literature in windscreen designs have been specific to their target applications introducing with them specific design challenges. Consider for example, wind farm noise measurements involve measuring large wind-turbines which produce higher acoustic emission levels in presence of wind. It is required that these measurements are performed outdoors, away from the source and in presence of a significant air flow. Various studies have suggested improved effectiveness in noise measurements with hollowed secondary windscreens [35] of larger sizes enclosing a layer of air to reduce air flow over the primary windscreens. Hansen et al. [47] compared the effectiveness of three radically different secondary windscreen designs - hemispherical, spherical and box buried into the earth. While the different designs showed varied degree of sound attenuation, measuring their insertion losses were instrumental. Novak et. al. [63] demonstrated the benefits of secondary windscreens over using just the primary, and compared the performance of commercially available secondary windscreen designs for wind-turbine measurements. The study cited lack of data with re-

gard to the insertion loss of the secondary windscreens to remark on the varied behavior of noise attenuation. More recently, similar studies were also performed in Germany by Kirsch [51] and Martens et al. [59] towards the development of a secondary windscreens as a remedy to improve signal-to-noise ratio in wind turbine measurements. As shown, there is significant interest in designing effective windscreens for outdoor applications but so far most reported studies followed a purely empirical design approach.

An ideal windscreen has negligible attenuation to source-induced perturbations and high attenuation to wind induced perturbations [25]. Towards that end, this study aims to provide a tool to compute the insertion loss of a windscreen and approach the ideal windscreen design from an *a priori* approach of characterization of materials.

3.2 Acoustic models

The problem can be mathematically formulated in various physically-relevant variables e.g., scalar fields like pressure, displacement potential or velocity potential; or vector fields like displacement or velocity, the choice often being the vector fields for coupled systems [21]. In this particular implementation, the time-harmonic acoustic oscillations are chosen to be represented by the displacement vector field involving the angular frequency ω , which will always be assumed positive. A sketch of the evaluated problem is shown in Figure 3.1.

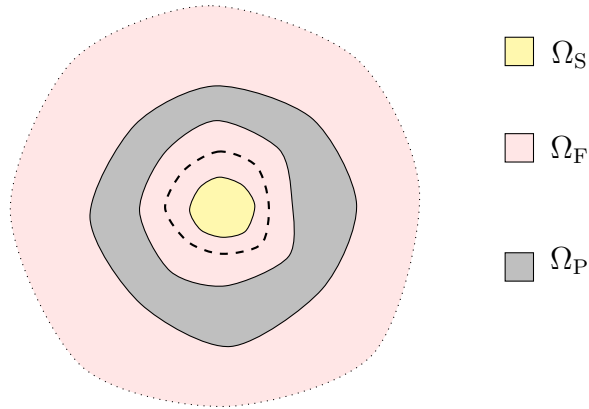


Figure 3.1: Schematic of the original problem configuration on unbounded domain

A series of time-harmonic acoustic models and associated assumptions are considered to arrive at a feasible overall numerical solution for the coupled problem involving a fluid model stated in unbounded domain Ω_F , a porous material with rigid solid frame located at Ω_P , and a steel microscopic wire mesh (highlighted with a dashed curve in Figure 3.1). This wire screen splits Ω_F in two disjoint parts: the inner and bounded fluid subdomain Ω_F^+ and the outer and unbounded fluid subdomain Ω_F^- . All these elements are surrounding a rigid solid structure Ω_S (highlighted in yellow in Figure 3.1), which is not part of the computational domain of the coupled problem since it is assumed rigid and hence its displacement is null.

Throughout this entire chapter we will assume that all the subdomains described above are open subsets of the three-dimensional space with piecewise smooth boundaries.

3.2.1 Fluid model stated in unbounded domains

The acoustic fluid is assumed to be homogeneous, non-viscous, compressible, isotropic and isentropic. A standard displacement-based model for the motion of this type of compressible fluids can be written in terms of a vector-valued governing equation which involves the mass density ρ_F , the sound speed c_F and whose primal unknown is the fluid displacement field $\mathbf{u}_F$. It is given by

$$-\operatorname{grad}(\rho_F c_F^2 \operatorname{div} \mathbf{u}_F) - \omega^2 \rho_F \mathbf{u}_F = \mathbf{f}_F \quad \text{in } \Omega_F^+ \cup \Omega_F^-, \quad (3.2.1)$$

where $\mathbf{f}_F$ is an external body force acting on the fluid domain. Once the displacement field is computed, the acoustic pressure can be computed using the compressible character of the acoustic fluid, this is,

$$p_F = -\rho_F c_F^2 \operatorname{div} \mathbf{u}_F.$$

In theory, the radiation condition at infinity is given by the so-called Sommerfeld radiation condition (written in terms of the pressure field). Equivalently, an analogous radiation condition can be written in terms of the displacement field (see Chapter 2), which gives the limiting condition at large distances,

$$\lim_{|\mathbf{x}| \rightarrow \infty} |\mathbf{x}| \left(\operatorname{div} \mathbf{u}_F - i \frac{\omega}{c_F} \mathbf{u}_F \cdot \frac{\mathbf{x}}{|\mathbf{x}|} \right) = 0. \quad (3.2.2)$$

3.2.2 Porous models of rigid solid frame

Porous materials come in various structural configuration like fibrous, foam or granular kinds, requiring different models to characterize their acoustic behaviour. A simplifying assumption to formulate a viable model is to consider the material to be isotropic and isothermal taking into account the macroscopic behaviour of the material. The structural frame of the layer is also considered rigid for further simplification allowing for an *fluid-equivalent* formulation written in terms of the macroscopic porous displacement field $\mathbf{u}_P$,

$$-\operatorname{grad}(K_P(\omega) \operatorname{div} \mathbf{u}_P) - \rho_P(\omega) \omega^2 \mathbf{u}_P = \mathbf{f}_P \quad \text{in } \Omega_P, \quad (3.2.3)$$

where $\rho_P(\omega)$ and $K_P(\omega)$ are the equivalent dynamic mass density and the equivalent dynamic bulk modulus of the porous material, respectively, and $\mathbf{f}_P$ is an external body force acting on the fluid domain. Once the porous displacement field is computed, the macroscopic acoustic pressure in the porous material can be computed using the compressible character of the acoustic fluid, this is,

$$p_P = -K_P(\omega) \operatorname{div} \mathbf{u}_P.$$

The material properties are to be determined either through experiments conducted *a priori* or through suitable models. A wide range of porous material models provide the material response along a range of frequencies. Among many other models, it can be cited: the Zwikker-Kosten model [76] based on one parameter (ϕ), the Delany-Bazley model [40] based on one parameter (σ), the Delany-Bazley-Miki model [61] based on one parameter (σ), the Darcy's like model [23, 73] based on two parameters (ϕ, σ), the general Miki model [61] based on four parameters ($\sigma, \phi, \alpha_\infty, M''$), the Attenborough model [8] based on four parameters ($\phi, \sigma, \alpha_\infty, M''$), the Wilson model [74] based on four parameters ($\rho_\infty, K_\infty, \tau_{\text{vor}}, \tau_{\text{ent}}$), the Johnson-Champoux-Allard model [32, 36] based on five parameters ($\sigma, \phi, \alpha_\infty, \Lambda, \Lambda'$), the Johnson-Champoux-Allard-Lafarge [32, 36, 52] model based on six parameters ($\sigma, \phi, \alpha_\infty, \Lambda, \Lambda', k'_0$), and the Johnson-Champoux-Allard-Pride-Lafarge [32, 36, 52, 65] model based on eight parameters ($\sigma, \phi, \alpha_\infty, \Lambda, \Lambda', k'_0, \alpha_0, \alpha'_0$). .

The fairly detailed six-parameter Johnson-Champoux-Allard-Lafarge (JCAL) model is chosen in the current article to obtain the dynamic porous mass density and bulk modulus, given by equations,

$$\rho_P(\omega) = \frac{\rho_F}{\phi} \alpha_\infty \left(1 - i \frac{\sigma \phi}{\omega \rho_F \alpha_\infty} \sqrt{1 + i \frac{4\alpha_\infty^2 \eta \rho_F \omega}{\sigma^2 \Lambda^2 \phi^2}} \right), \quad (3.2.4)$$

$$K_P(\omega) = \frac{\gamma P_F / \phi}{\gamma - (\gamma - 1) \left(1 - i \frac{\eta \phi}{\rho_F k'_0 \omega \text{Pr}} \sqrt{1 + i \frac{4k'_0{}^2 \rho_F \omega \text{Pr}}{\eta \Lambda'^2 \phi^2}} \right)^{-1}}. \quad (3.2.5)$$

The JCAL model is reliable for porous materials with arbitrarily shaped pores. The parameters in the model: porosity ϕ , flow resistivity σ , tortuosity α_∞ , viscous characteristic length Λ , thermal characteristic length Λ' and static thermal permeability k'_0 ; effectively capture the macroscopic thermal, viscous and inertial characteristics of the porous material. The fluid state properties like fluid mass density ρ_F , specific heat ratio γ , Prandtl number Pr , and equilibrium fluid pressure P_F encapsulate the properties of the saturating fluid.

Coupling conditions between the fluid and porous subdomains ensure the kinetic and the kinematic assumptions about the mechanical behaviour of the common boundary between fluid and porous materials. In the case of a perfect coupling interface boundary in absence of external surface loads and mechanical dissipation effects, it is assumed the continuity of the normal component of the displacement field (kinematic) and no jumps are present in the pressure fields associated with the fluid and porous displacements. Hence, these assumptions yield to

$$\mathbf{u}_F \cdot \mathbf{n} = \mathbf{u}_P \cdot \mathbf{n}, \quad (3.2.6)$$

and, taking into account that $p_F = p_P$, also

$$-\rho_F c_F^2 \text{div } \mathbf{u}_F = -K_P(\omega) \text{div } \mathbf{u}_P, \quad (3.2.7)$$

on the coupling interface between both fluid and porous subdomains. Notice that $\mathbf{n}$ denotes the unit normal vector on the coupling interface between the fluid and the porous subdomains.

3.2.3 Steel Mesh Model

The acoustic sensors are generally housed in a wire mesh of sub-millimetre thickness for protection. This layer of negligible thickness impacts the acoustic measurements in lower frequencies (see Figure 3.2).



Figure 3.2: Steel mesh used in engineering applications. (a) PU Regular Probe covered in steel mesh, (b) Microscopic image of the steel mesh.

To account for the effects of a wire mesh on the acoustic quantities of interest (such as the displacement, the velocity or the intensity fields), they are modeled as an interface with a wall-impedance condition across the contact surface [13], i.e.,

$$\mathbf{u}_F^+ \cdot \mathbf{n} = \mathbf{u}_F^- \cdot \mathbf{n}, \quad (3.2.8)$$

$$-\rho_F c^2 \operatorname{div} \mathbf{u}_F^+ + \rho_F c^2 \operatorname{div} \mathbf{u}_F^- = -i\omega Z(\omega) \mathbf{u}_F \cdot \mathbf{n}. \quad (3.2.9)$$

Here, the superscript signs $\pm$ denote the restriction of the displacement field $\mathbf{u}_F$ to Ω_F^+ and Ω_F^- , the inner or the outer part of the fluid domain, which are separated by the wire screen, respectively.

Equation (3.2.8) and (3.2.9) implies a continuity along the normal displacement field and a jump in the pressure field across the contact surface. Computing the wall-impedance coefficient $Z(\omega)$ can be done according to various studies suggested by various studies like by Zwikker-Kosten [77], Atalla [7] and Maa [55, 57] for different kinds of wire meshes. An initial assumption is to approximate the steel mesh to be with cylindrical pores and account for the sound transmission using the theory of flow through short tubes. Using this simplified approach, Maa [55] provides the following expression for the complex-valued impedance:

$$Z(\omega) = \frac{32\eta t}{pd^2} \left(\sqrt{1 + \frac{s^2}{32}} + \sqrt{\frac{sd}{4t}} \right) + i \frac{\rho \omega t}{p} \left(1 + \frac{1}{\sqrt{9 + s^2/2}} + \frac{0.85d}{t} \right), \quad (3.2.10)$$

where $s = \sqrt{\rho\omega/(4\eta)}$ is a non-dimensional parameter, which depends on the diameter of the perforation holes d , the angular frequency ω , the dynamic fluid viscosity η , and the fluid mass density ρ_F . The complex-valued expression of the surface impedance associated with the steel mesh also depends on the perforation ratio p , the diameter d of the perforation in the mesh, and the overall thickness t of the steel mesh.

3.3 Scattering Problem

The governing motion equations written in the section above are written in terms of the total displacement field, this is, taking into account all the motion perturbations produced by the time-harmonic acoustic wave propagation throughout the different material subdomains.

However, in typical scattering problem it is assumed that the total acoustic fields (such as the displacement of the pressure field) are the addition of two different contributions: an *a priori* known impinging wave, given by a wave solution in the fluid domain (for instance, $\mathbf{u}^{\text{inc}} = \mathbf{u}_0 \exp(i\mathbf{k}_F \cdot \mathbf{x})$, with $|\mathbf{k}_F| = \omega/c_F$ and $\mathbf{k}_F$ is parallel to $\mathbf{u}_0$) and a second term, which accounts for the diffracted acoustic effects due to the reflections of the incident wave on the mechanical structure of the problem. The displacement fields related to the scattering contribution will be denoted with the superscript $(\)^{\text{sc}}$.

Hence, the entire formulation for the coupled scattering problem is given by the following system of equations: *Given a fixed angular frequency value $\omega > 0$ and incident field $\mathbf{u}^{\text{inc}}$, find the $\mathbf{u}_F = \mathbf{u}^{\text{inc}} + \mathbf{u}_F^{\text{sc}}$, $\mathbf{u}_P = \mathbf{u}^{\text{inc}} + \mathbf{u}_P^{\text{sc}}$ such that*

$$-\text{grad}(\rho_F c_F^2 \text{div } \mathbf{u}_F) - \rho_F \omega^2 \mathbf{u}_F = \mathbf{0} \quad \text{in } \Omega_F^+ \cup \Omega_F^-, \quad (3.3.1a)$$

$$-\text{grad}(K_P(\omega) \text{div } \mathbf{u}_P) - \rho_P(\omega) \omega^2 \mathbf{u}_P = \mathbf{0} \quad \text{in } \Omega_P, \quad (3.3.1b)$$

$$\mathbf{u}_F \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_S, \quad (3.3.1c)$$

$$\mathbf{u}_F^+ \cdot \mathbf{n} - \mathbf{u}_F^- \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_M, \quad (3.3.1d)$$

$$\rho_F c_F^2 \text{div } \mathbf{u}_F^+ - \rho_F c_F^2 \text{div } \mathbf{u}_F^- - i\omega Z(\omega) \mathbf{u}_F \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_M, \quad (3.3.1e)$$

$$\mathbf{u}_F \cdot \mathbf{n} - \mathbf{u}_P \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_C, \quad (3.3.1f)$$

$$\rho_F c_F^2 \text{div } \mathbf{u}_F - K_P(\omega) \text{div } \mathbf{u}_P = 0 \quad \text{on } \Gamma_C, \quad (3.3.1g)$$

$$\lim_{|\mathbf{x}| \rightarrow \infty} |\mathbf{x}| \left(\text{div } \mathbf{u}_F^{\text{sc}} - i \frac{\omega}{c_F} \mathbf{u}_F^{\text{sc}} \cdot \frac{\mathbf{x}}{|\mathbf{x}|} \right) = 0, \quad (3.3.1h)$$

where Γ_S is the boundary of the mechanics structure surrounded by the fluid subdomain, Γ_C it the coupling interface between the fluid and the porous subdomains, Γ_M it the coupling interface between the inner and outer fluid subdomains across the steel wire mesh. Notice that we have abused on the notation and all the occurrences of $\mathbf{n}$ accounts for the unit normal vector on any boundary involved in the coupling problem.

The coupling between different layers occurs only at interfaces, and the scattered displacement vector fields, $\mathbf{u}_F$, $\mathbf{u}_P$ and $\mathbf{u}_{\text{PML}}$, are defined in exclusive domains where all the boundaries assumed smooth.

3.3.1 Perfectly Matched Layers

The unbounded domain of the acoustic fluid implies a complete radiation behaviour of all outgoing scattered fields. However, since an actual computation needs a finite domain, the outer boundary needs to absorb any outward incident waves. With this aim, the Cartesian Perfectly Matched Layer technique [11] will be introduced (see Figure 3.3).

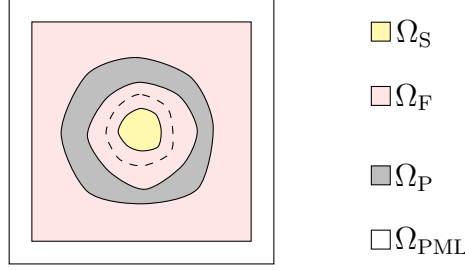


Figure 3.3: Schematic of the domain with the perfectly matched layer

Since the Sommerfeld radiation condition (3.2.2) is realistically hard to impose and is typically accomplished by a "sponge" layer with dissipating properties known as the Perfectly Matched Layer (PML). This is achieved by a Helmholtz-like governing equation with auxiliary stretching factors in the governing equation within the PML (see [16] and Chapter 2 for a detailed derivation),

$$-\text{grad}(\rho_F c_F^2 (H^{-1} : \text{grad } \mathbf{u}_{\text{PML}})) - \rho_F \omega^2 H \mathbf{u}_{\text{PML}} = \mathbf{0}. \quad (3.3.2)$$

where

$$H = \sum_{j=1}^3 \gamma_j \mathbf{e}_j \otimes \mathbf{e}_j,$$

with $\mathbf{e}_j$, $1 \leq j \leq 3$, being the canonical vector basis associated with the Cartesian system of reference. The optimally-tuned functions provided by Bermudez et al. [24] are chosen among the various choices for the complex stretching functions γ_j , giving,

$$\gamma_j(x_j) = 1 + i \frac{c_F}{\omega(L_j^* - |x_j|)} \quad \text{for } L_j \leq |x_j| \leq L_j^*. \quad (3.3.3)$$

Here, L_j and L_j^* are respectively the lengths of the Cartesian box of the truncated fluid domain and the PML domain, along the direction x_j from the origin. The piece-wise definition of γ_j ensures the absorption of waves only along the outward direction of propagation. The interface conditions between the PML and fluid subdomains are just the continuity in the normal component of the scattered displacement and pressure fields:

$$\mathbf{u}_F^{\text{sc}} \cdot \mathbf{n} = \mathbf{u}_{\text{PML}} \cdot \mathbf{n}, \quad (3.3.4)$$

and

$$-\rho_F c_F^2 \text{div } \mathbf{u}_F^{\text{sc}} = -\rho_F c_F^2 H^{-1} : \text{grad } \mathbf{u}_{\text{PML}}. \quad (3.3.5)$$

Hence, the entire formulation for the coupled scattering problem is given by the following system of equations: *Given a fixed angular frequency value $\omega > 0$ and incident field $\mathbf{u}^{\text{inc}}$,*

find the $\mathbf{u}_F = \mathbf{u}^{\text{inc}} + \mathbf{u}_F^{\text{sc}}$, $\mathbf{u}_P = \mathbf{u}^{\text{inc}} + \mathbf{u}_P^{\text{sc}}$, and $\mathbf{u}_{\text{PML}}$ such that

$$-\text{grad}(\rho_F c_F^2 \text{div } \mathbf{u}_F) - \rho_F \omega^2 \mathbf{u}_F = \mathbf{0} \quad \text{in } \Omega_F^+ \cup \Omega_F^-, \quad (3.3.6a)$$

$$-\text{grad}(K_P(\omega) \text{div } \mathbf{u}_P) - \rho_P(\omega) \omega^2 \mathbf{u}_P = \mathbf{0} \quad \text{in } \Omega_P, \quad (3.3.6b)$$

$$-\text{grad}(\rho_F c_F^2 H^{-1} : \text{grad } \mathbf{u}_{\text{PML}}) - \rho_F \omega^2 H \mathbf{u}_{\text{PML}} = \mathbf{0} \quad \text{in } \Omega_{\text{PML}}, \quad (3.3.6c)$$

$$\mathbf{u}_F \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_S, \quad (3.3.6d)$$

$$\mathbf{u}_F^+ \cdot \mathbf{n} - \mathbf{u}_F^- \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_M, \quad (3.3.6e)$$

$$\rho_F c^2 \text{div } \mathbf{u}_F^+ - \rho_F c^2 \text{div } \mathbf{u}_F^- - i\omega Z(\omega) \mathbf{u}_F \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_M, \quad (3.3.6f)$$

$$\mathbf{u}_F \cdot \mathbf{n} - \mathbf{u}_P \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_C, \quad (3.3.6g)$$

$$\rho_F c_F^2 \text{div } \mathbf{u}_F - K_P(\omega) \text{div } \mathbf{u}_P = 0 \quad \text{on } \Gamma_C, \quad (3.3.6h)$$

$$\mathbf{u}_F^{\text{sc}} \cdot \mathbf{n} - \mathbf{u}_{\text{PML}} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{\text{PML}}, \quad (3.3.6i)$$

$$\rho_F c_F^2 \text{div } \mathbf{u}_F^{\text{sc}} - \rho_F c_F^2 H^{-1} : \text{grad } \mathbf{u}_{\text{PML}} = 0 \quad \text{on } \Gamma_{\text{PML}}, \quad (3.3.6j)$$

$$\mathbf{u}_{\text{PML}} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_\star, \quad (3.3.6k)$$

where Γ_{PML} is the coupling interface between outer fluid subdomains and the PML layer, and $\Gamma_\star$ is the outer boundary of the PML layer.

The coupled system (3.3.6) has been written in terms of the total and the scattered fields. However, for variational purposes it is much more convenient to write it entirely in terms of the scattered fields. Straightforward computations leads to the following equivalent problem: *Given a fixed angular frequency value $\omega > 0$ and incident field $\mathbf{u}^{\text{inc}}$, find the $\mathbf{u}_F^{\text{sc}}$, $\mathbf{u}_P^{\text{sc}}$, and $\mathbf{u}_{\text{PML}}$ such that*

$$-\text{grad}(\rho_F c_F^2 \text{div } \mathbf{u}_F^{\text{sc}}) - \rho_F \omega^2 \mathbf{u}_F^{\text{sc}} = \mathbf{0} \quad \text{in } \Omega_F^\pm, \quad (3.3.7a)$$

$$-\text{grad}(K_P(\omega) \text{div } \mathbf{u}_P^{\text{sc}}) - \rho_P(\omega) \omega^2 \mathbf{u}_P^{\text{sc}} = \mathbf{f}_P^{\text{inc}} \quad \text{in } \Omega_P, \quad (3.3.7b)$$

$$-\text{grad}(\rho_F c_F^2 H^{-1} : \text{grad } \mathbf{u}_{\text{PML}}) - \rho_F \omega^2 H \mathbf{u}_{\text{PML}} = \mathbf{0} \quad \text{in } \Omega_{\text{PML}}, \quad (3.3.7c)$$

$$\mathbf{u}_F^{\text{sc}} \cdot \mathbf{n} = -\mathbf{u}^{\text{inc}} \cdot \mathbf{n} \quad \text{on } \Gamma_S, \quad (3.3.7d)$$

$$\mathbf{u}_F^{+, \text{sc}} \cdot \mathbf{n} - \mathbf{u}_F^{-, \text{sc}} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_M, \quad (3.3.7e)$$

$$-\rho_F c^2 (\text{div } \mathbf{u}_F^{+, \text{sc}} - \text{div } \mathbf{u}_F^{-, \text{sc}}) - i\omega Z(\omega) \mathbf{u}_F^{\text{sc}} \cdot \mathbf{n} = i\omega Z(\omega) \mathbf{u}^{\text{inc}} \cdot \mathbf{n} \quad \text{on } \Gamma_M, \quad (3.3.7f)$$

$$\mathbf{u}_F^{\text{sc}} \cdot \mathbf{n} - \mathbf{u}_P^{\text{sc}} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_C, \quad (3.3.7g)$$

$$\rho_F c_F^2 \text{div } \mathbf{u}_F^{\text{sc}} - K_P(\omega) \text{div } \mathbf{u}_P^{\text{sc}} = g_P^{\text{inc}} \quad \text{on } \Gamma_C, \quad (3.3.7h)$$

$$\mathbf{u}_F^{\text{sc}} \cdot \mathbf{n} - \mathbf{u}_{\text{PML}} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{\text{PML}}, \quad (3.3.7i)$$

$$\rho_F c_F^2 \text{div } \mathbf{u}_F^{\text{sc}} - \rho_F c_F^2 H^{-1} : \text{grad } \mathbf{u}_{\text{PML}} = 0 \quad \text{on } \Gamma_{\text{PML}}, \quad (3.3.7j)$$

$$\mathbf{u}_{\text{PML}} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_\star, \quad (3.3.7k)$$

where the new non-null source terms due to the incident wave are given by

$$\mathbf{f}_P^{\text{inc}} = \text{grad}(K_P(\omega) \text{div } \mathbf{u}^{\text{inc}}) + \rho_P(\omega) \omega^2 \mathbf{u}^{\text{inc}} \quad \text{in } \Omega_P, \quad (3.3.8)$$

$$g_P^{\text{inc}} = -(\rho_F c_F^2 - K_P(\omega)) \text{div } \mathbf{u}^{\text{inc}} \quad \text{in } \Gamma_C. \quad (3.3.9)$$

3.4 Variational formulation

The first step to write the variational formulation associated with the coupled system (3.3.6) consist in the derivation of a weak problem for each of the partial differential equations which govern the motion in the fluid, porous and PML subdomains. With that purpose, we formally multiply the fluid governing equation (3.3.7a) by a test function $\mathbf{v}_F$ and using a Green's identity in Ω_F^+ and Ω_F^- , we have

$$\begin{aligned} & \int_{\Omega_F^+ \cup \Omega_F^-} \rho_F c_F^2 \operatorname{div} \mathbf{u}_F^{\text{sc}} \operatorname{div} \mathbf{v}_F dV - \int_{\Omega_F^+ \cup \Omega_F^-} \rho_F \omega^2 \mathbf{u}_F^{\text{sc}} \cdot \mathbf{v}_F dV \\ & - \int_{\Gamma_S \cup \Gamma_{\text{PML}} \cup \Gamma_C} \rho_F c_F^2 \operatorname{div} \mathbf{u}_F^{\text{sc}} (\mathbf{v}_F \cdot \mathbf{n}) dS - \int_{\Gamma_M} \rho_F c_F^2 (\operatorname{div} \mathbf{u}_F^{+, \text{sc}} - \operatorname{div} \mathbf{u}_F^{-, \text{sc}}) \mathbf{v}_F \cdot \mathbf{n} dS = 0. \end{aligned} \quad (3.4.1)$$

Analogously, using the same procedure for the governing equation in the porous domain (3.3.7b) with a test function $\mathbf{v}_P$, it holds

$$\begin{aligned} & \int_{\Omega_P} K_P(\omega) \operatorname{div} \mathbf{u}_P^{\text{sc}} \operatorname{div} \mathbf{v}_P dV - \int_{\Omega_P} \rho_P(\omega) \omega^2 \mathbf{u}_P^{\text{sc}} \cdot \mathbf{v}_P dV \\ & - \int_{\Gamma_C} K_P(\omega) \operatorname{div} \mathbf{u}_P^{\text{sc}} (\mathbf{v}_P \cdot \mathbf{n}) dS = \int_{\Omega_P} \mathbf{f}_P^{\text{inc}} \cdot \mathbf{v}_P dV. \end{aligned} \quad (3.4.2)$$

Similarly, multiplying the motion equation (3.3.7c) in the PML subdomain by a test function $\mathbf{v}_{\text{PML}}$ and using a Green's identity, we obtain

$$\begin{aligned} & \int_{\Omega_{\text{PML}}} \rho_F c_F^2 (H^{-1} : \operatorname{grad} \mathbf{u}_{\text{PML}}) \operatorname{div} \mathbf{v}_{\text{PML}} dV - \int_{\Omega_{\text{PML}}} \rho_F \omega^2 H \mathbf{u}_{\text{PML}} \cdot \mathbf{v}_{\text{PML}} dV \\ & - \int_{\Gamma_{\text{PML}} \cup \Gamma_\star} \rho_F c_F^2 (H^{-1} : \operatorname{grad} \mathbf{u}_{\text{PML}}) (\mathbf{v}_{\text{PML}} \cdot \mathbf{n}) dS = 0. \end{aligned} \quad (3.4.3)$$

Adding (3.4.1), (3.4.2), and (3.4.3), and taking into account the coupling conditions (3.3.7d)-(3.3.7k), and we assume that the test functions also verify the same coupling and boundary conditions with null data, we arrive to the equality

$$a((\mathbf{u}_F^{\text{sc}}, \mathbf{u}_P^{\text{sc}}, \mathbf{u}_{\text{PML}}), (\mathbf{v}_F, \mathbf{v}_P, \mathbf{v}_{\text{PML}})) = b(\mathbf{v}_F, \mathbf{v}_P, \mathbf{v}_{\text{PML}}),$$

where

$$\begin{aligned}
a((\mathbf{u}_F^{\text{sc}}, \mathbf{u}_P^{\text{sc}}, \mathbf{u}_{\text{PML}}), (\mathbf{v}_F, \mathbf{v}_P, \mathbf{v}_{\text{PML}})) &= \int_{\Omega_F^+ \cup \Omega_F^-} \rho_F c_F^2 \operatorname{div} \mathbf{u}_F^{\text{sc}} \operatorname{div} \mathbf{v}_F dV - \int_{\Omega_F^+ \cup \Omega_F^-} \rho_F \omega^2 \mathbf{u}_F^{\text{sc}} \cdot \mathbf{v}_F dV \\
&+ \int_{\Omega_P} K_P(\omega) \operatorname{div} \mathbf{u}_P^{\text{sc}} \operatorname{div} \mathbf{v}_P dV - \int_{\Omega_P} \rho_P(\omega) \omega^2 \mathbf{u}_P^{\text{sc}} \cdot \mathbf{v}_P dV \\
&+ \int_{\Omega_{\text{PML}}} \rho_F c_F^2 (H^{-1} : \operatorname{grad} \mathbf{u}_{\text{PML}}) \operatorname{div} \mathbf{v}_{\text{PML}} dV - \int_{\Omega_{\text{PML}}} \rho_F \omega^2 H \mathbf{u}_{\text{PML}} \cdot \mathbf{v}_{\text{PML}} dV \\
&+ \int_{\Gamma_M} i\omega Z(\omega) (\mathbf{u}_F^{\text{sc}} \cdot \mathbf{n}) (\mathbf{v} \cdot \mathbf{n}) dS, \quad (3.4.4)
\end{aligned}$$

and

$$\begin{aligned}
b^{\text{inc}}(\mathbf{v}_F, \mathbf{v}_P, \mathbf{v}_{\text{PML}}) &= \int_{\Omega_P} \mathbf{f}_P^{\text{inc}} \cdot \mathbf{v}_P dV + \int_{\Gamma_C} g_P^{\text{inc}} \mathbf{v}_P \cdot \mathbf{n} dS \\
&- \int_{\Gamma_M} i\omega Z(\omega) (\mathbf{u}^{\text{inc}} \cdot \mathbf{n}) (\mathbf{v} \cdot \mathbf{n}) dS. \quad (3.4.5)
\end{aligned}$$

Due to the expressions involved in the bilinear form a , a suitable choice of the functional space V associated with the displacement fields involved in the couple problem (3.3.7) is defined by

$$\begin{aligned}
V = \{ &(\mathbf{v}_F, \mathbf{v}_P, \mathbf{v}_{\text{PML}}) \in [L^2(\Omega_F)]^3 \times [L^2(\Omega_P)]^3 \times [L^2(\Omega_{\text{PML}})]^3 : \mathbf{v}_F \in \mathbf{H}(\operatorname{div}, \Omega_F), \\
&\mathbf{v}_P \in \mathbf{H}(\operatorname{div}, \Omega_P), H \mathbf{v}_{\text{PML}} \in [L^2(\Omega_{\text{PML}})]^3, H^{-1} : \operatorname{grad} \mathbf{v}_{\text{PML}} \in L^2(\Omega_{\text{PML}}), \\
&\mathbf{v}_F \cdot \mathbf{n} = \mathbf{v}_P \cdot \mathbf{n} \text{ on } \Gamma_C, \mathbf{v}_F \cdot \mathbf{n} = \mathbf{v}_{\text{PML}} \cdot \mathbf{n} \text{ on } \Gamma_{\text{PML}}, \mathbf{v}_{\text{PML}} \cdot \mathbf{n} = 0 \text{ on } \Gamma_\star \}. \quad (3.4.6)
\end{aligned}$$

Hence, the variational form of the coupled problem (3.3.7) is described as follows: *Given a fixed angular frequency value $\omega > 0$ and incident field $\mathbf{u}^{\text{inc}}$, find $(\mathbf{u}_F^{\text{sc}}, \mathbf{u}_P^{\text{sc}}, \mathbf{u}_{\text{PML}}) \in V$ such that $\mathbf{u}_F \cdot \mathbf{n} = -\mathbf{u}^{\text{inc}} \cdot \mathbf{n}$ on Γ_S and satisfying*

$$a((\mathbf{u}_F^{\text{sc}}, \mathbf{u}_P^{\text{sc}}, \mathbf{u}_{\text{PML}}), (\mathbf{v}_F, \mathbf{v}_P, \mathbf{v}_{\text{PML}})) = b^{\text{inc}}(\mathbf{v}_F, \mathbf{v}_P, \mathbf{v}_{\text{PML}}) \quad (3.4.7)$$

for all test functions $(\mathbf{v}_F, \mathbf{v}_P, \mathbf{v}_{\text{PML}}) \in V$ such that $\mathbf{v}_F \cdot \mathbf{n} = 0$ on Γ_S .

The previous variational formulation (3.4.7) can be now rewriting in a simple manner since the unknown field of the model, this is, the scattering displacement field $\mathbf{u}$ could be defined within the entire computational domain $\Omega = \operatorname{int}(\overline{\Omega}_F \cup \overline{\Omega}_P \cup \overline{\Omega}_{\text{PML}})$ by means of the following piecewise definition:

$$\mathbf{u} = \begin{cases} \mathbf{u}_F^{+, \text{sc}} & \text{in } \Omega_F^+, \\ \mathbf{u}_F^{-, \text{sc}} & \text{in } \Omega_F^-, \\ \mathbf{u}_P^{\text{sc}} & \text{in } \Omega_P, \\ \mathbf{u}_{\text{PML}} & \text{in } \Omega_{\text{PML}}. \end{cases} \quad (3.4.8)$$

Since the normal component of the displacement field is continuous across the coupling boundaries (see coupling conditions (3.3.7e), (3.3.7g), and (3.3.7j)) and the tensor H is locally bounded in a neighbourhood of Γ_{PML} it is possible to show that $\mathbf{u}$ belongs to $H(\widetilde{\text{div}}, \Omega)$, where the differential operator $\widetilde{\text{div}}$ coincides with the standard divergence operator in the physical (fluid and porous) subdomains whereas $\widetilde{\text{div}}\boldsymbol{\psi} = H^{-1} : \text{grad } \boldsymbol{\psi}$ for any smooth test function $\boldsymbol{\psi}$. In this manner, taking into account (3.4.8), the statement $\mathbf{u} \in H(\widetilde{\text{div}}, \Omega)$ is equivalent to $(\mathbf{u}_{\text{F}}^{\text{sc}}, \mathbf{u}_{\text{P}}^{\text{sc}}, \mathbf{u}_{\text{PML}}) \in V$.

3.5 Finite element discretization

The discretization procedure for solving the variational problem (3.4.7) is initiated by partitioning the computational domain Ω into a tetrahedrons to create an approximate tetrahedral mesh, $\Omega = \cup_{T \in \mathcal{T}_h} T$ and such that it is conformal with the boundary partition and the interface boundaries Γ_{C} , Γ_{M} , and Γ_{PML} between the fluid subdomain, the steel wire mesh, and the porous and PML subdomains.

To discretize the function space V by considering this tetrahedral mesh $\mathcal{T}_h$, first-order Raviart-Thomas Finite Elements are used. Hence, using the global definition of the displacement field (3.4.8), the discrete finite element space is defined by

$$V_h = \left\{ \mathbf{w}_h \in \mathbf{H}(\widetilde{\text{div}}, \Omega) : \mathbf{w}_h|_T = \mathbf{a} + b\mathbf{x}, \mathbf{a} \in \mathbb{C}^3, b \in \mathbb{C}, T \in \mathcal{T}_h, \right. \\ \left. \mathbf{w}_h \cdot \mathbf{n} = 0 \text{ on } \Gamma_{\star} \right\}. \quad (3.5.1)$$

Hence, if we consider $\mathbf{u}_h^{\text{inc}}$ the interpolant of $\mathbf{u}^{\text{inc}}$ in the Raviart-Thomas discrete space, the discrete variational problem associated with the coupled problem (3.3.7) is described as follows: *Given a fixed angular frequency value $\omega > 0$ and a discrete incident field $\mathbf{u}_h^{\text{inc}}$, find $\mathbf{u}_h \in V_h$ such that $\mathbf{u}_h \cdot \mathbf{n} = -\mathbf{u}_h^{\text{inc}} \cdot \mathbf{n}$ on Γ_{S} and satisfying*

$$a((\mathbf{u}_h|_{\Omega_{\text{F}}}, \mathbf{u}_h|_{\Omega_{\text{P}}}, \mathbf{u}_h|_{\Omega_{\text{PML}}}), (\mathbf{v}_h|_{\Omega_{\text{F}}}, \mathbf{v}_h|_{\Omega_{\text{P}}}, \mathbf{v}_h|_{\Omega_{\text{PML}}})) = b^{\text{inc}}(\mathbf{v}_h|_{\Omega_{\text{F}}}, \mathbf{v}_h|_{\Omega_{\text{P}}}, \mathbf{v}_h|_{\Omega_{\text{PML}}}) \quad (3.5.2)$$

for all test functions $\mathbf{v}_h \in V_h$ such that $\mathbf{v}_h \cdot \mathbf{n} = 0$ on Γ_{S} .

The Raviart-Thomas basis elements are defined from the normal component of the discrete displacement approximations on the face normals of tetrahedra (see Figure 3.4). They are particularly chosen for their characteristically constant divergence on each tetrahedral element. In that manner, let us define the basis of V_h , denoted by $\{\boldsymbol{\varphi}_j\}_{j=1}^n$, such that

$$\boldsymbol{\varphi}_j \cdot \mathbf{n}|_{F_i} = \delta_{ij} \quad 1 \leq i, j \leq n,$$

where n is the total number of faces in the mesh $\mathcal{T}_h$, F_i is the i -th face of the tetrahedral mesh, and δ_{ij} is the Kronecker's delta. So, any function pair can be represented in terms of this basis as $\mathbf{w}_h = \sum_{j=1}^n w_j \boldsymbol{\varphi}_j$. The vector containing the linear coefficients w_j of $\mathbf{w}_h$ is denoted by $\vec{w}_h$.

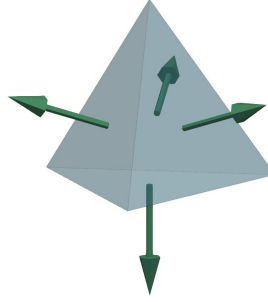


Figure 3.4: The Raviart Thomas Finite Element on a tetrahedron. The arrows denote the normal vectors of each face, where the degrees of freedom are symbolically located.

Finally, the discrete variational problem (3.5.2) can be written in matrix form as follows:

$$(\mathcal{K}_F + \mathcal{K}_P(\omega) + \mathcal{K}_{PML}(\omega)) + i\omega Z(\omega)\mathcal{C}_M - \omega^2(\mathcal{M}_F + \mathcal{M}_P(\omega) + \mathcal{M}_{PML}(\omega))\vec{w}_h = \vec{f}_h(\omega),$$

where each matrix of the discrete system described above are defined by

$$\begin{aligned} [\mathcal{K}_F]_{ij} &= \int_{\Omega_F} \rho_F c_F^2 \operatorname{div} \varphi_i \operatorname{div} \varphi_j \, dV & 1 \leq i, j \leq n, \\ [\mathcal{K}_P]_{ij} &= \int_{\Omega_P} K_P(\omega) \operatorname{div} \varphi_i \operatorname{div} \varphi_j \, dV & 1 \leq i, j \leq n, \\ [\mathcal{K}_{PML}]_{ij} &= \int_{\Omega_{PML}} \rho_F c_F^2 (H^{-1} : \operatorname{grad} \varphi_i) \operatorname{div} \varphi_j \, dV & 1 \leq i, j \leq n, \\ [\mathcal{C}_M]_{ij} &= \int_{\Gamma_M} (\varphi_i \cdot \mathbf{n})(\varphi_j \cdot \mathbf{n}) \, dS & 1 \leq i, j \leq n, \\ [\mathcal{M}_F]_{ij} &= \int_{\Omega_F} \rho_F \varphi_i \cdot \varphi_j \, dV & 1 \leq i, j \leq n, \\ [\mathcal{M}_P]_{ij} &= \int_{\Omega_P} \rho_P(\omega) \varphi_i \cdot \varphi_j \, dV & 1 \leq i, j \leq n, \\ [\mathcal{M}_{PML}]_{ij} &= \int_{\Omega_{PML}} \rho_F H \varphi_i \cdot \varphi_j \, dV & 1 \leq i, j \leq n, \end{aligned}$$

and the right-hand side

$$[\vec{f}_h]_i = \int_{\Omega_P} \mathbf{f}_h^{\text{inc}} \cdot \mathbf{v}_P \, dV + \int_{\Gamma_C} g_h^{\text{inc}} \mathbf{v}_P \cdot \mathbf{n} - \int_{\Gamma_M} i\omega Z(\omega) (\mathbf{u}_h^{\text{inc}} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n}) \, dS,$$

where $\mathbf{f}_h^{\text{inc}}$ and g_h^{inc} are suitable discrete interpolants of source functions $\mathbf{f}_P^{\text{inc}}$ and g_P^{inc} , respectively.

A solution may then be obtained by solving this linear system of equations. The following sections describe the implementation of this model along with a specific example of acoustic transmission across a porous layer around a vibrating sphere.

3.6 Numerical validation

A sphere of radius R_0 is placed at the origin within an acoustic fluid of density ρ_F . It is enclosed in a hollow spherical porous disk with an inner radius of R_1 and an outer radius of R_2 (see Figure 3.5). Given that the surface of the sphere vibrates at a constant rate producing oscillations of frequency ω , the problem is then to compute the scattered field (reflected and transmitted to the fluid subdomain) due to the porous layer.

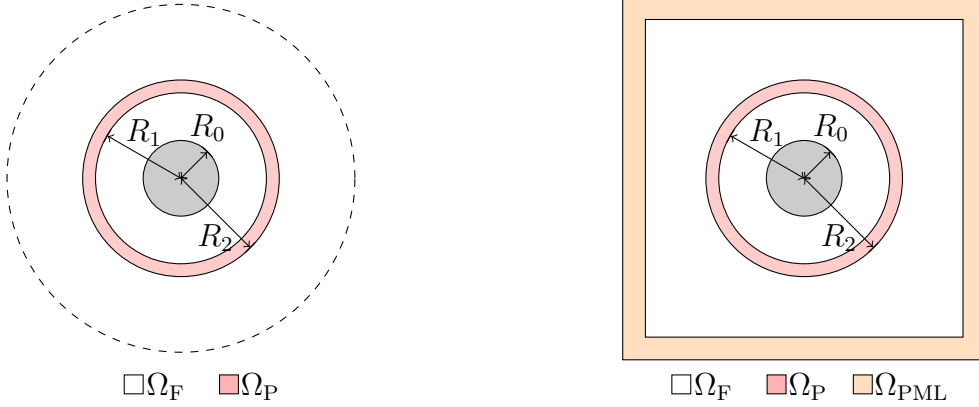


Figure 3.5: Vertical two-dimensional section of the schematic view of the spherical test case posed in an unbounded domain (left) and truncated using Cartesian Perfectly Matched Layers (right).

Consequently, the scattering problem is stated in the fluid and porous subdomains given by

$$\begin{aligned}\Omega_F &= \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : R_0 < |\mathbf{x}| < R_1 \text{ or } R_2 < |\mathbf{x}|\}, \\ \Omega_P &= \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : R_1 < |\mathbf{x}| < R_2\}.\end{aligned}$$

The incident time-harmonic wave is a radial wave

$$\begin{aligned}\mathbf{u}^{\text{inc}}(\mathbf{x}) &= \frac{1}{\omega^2 \rho_F} \text{grad } p^{\text{inc}} = U_0 \left(ik - \frac{1}{|\mathbf{x}|} \right) \frac{e^{ik_F |\mathbf{x}|}}{|\mathbf{x}|} \mathbf{e}_r, \\ p^{\text{inc}}(\mathbf{x}) &= P_0 \frac{e^{ik_F |\mathbf{x}|}}{|\mathbf{x}|},\end{aligned}$$

where $k_F = \omega/c_F$ is the wavenumber in the fluid subdomain and $U_0 = P_0/(\omega^2 \rho_F)$. Due to the spherical symmetry of the physical subdomains and the incident field, it is straightforward to check that the exact solution depends only on the radial coordinate $r = |\mathbf{x}|$.

For simplicity on the computations, and since the original scattering problem can be rewritten in terms of the scalar pressure field, the exact solution is computed by enforcing a constant Neumann boundary condition on the surface of the sphere (the value given by the trace of the pressure field on the boundary Γ_S located at $r = R_0$). Due to the radial symmetry of the problem, it is ensured that the Helmholtz equation has only radially dependent

solutions which may be expressed as a linear combination of incoming and outgoing radial waves. Writing in terms of the scattered pressure fields, the exact solution is given by

$$p(r) = \begin{cases} A_1 \frac{\exp(ik_F r)}{r} + B_1 \frac{\exp(-ik_F r)}{r}, & r \in (R_0, R_1), \\ A_2 \frac{\exp(ik_P r)}{r} + B_2 \frac{\exp(-ik_P r)}{r}, & r \in (R_1, R_2), \\ B_3 \frac{\exp(-ik_F r)}{r}, & r \in (R_2, \infty), \end{cases} \quad (3.6.1)$$

where the $k_F = \omega/c_F$ and $k_P = \omega\sqrt{\rho_P(\omega)/K_P(\omega)}$ are the wavenumbers of the fluid and porous models, respectively. The coefficients A_1 and A_2 correspond to the coefficients of incoming waves, and B_1 , B_2 , and B_3 to the coefficients of outgoing waves. In an unbounded domain, since there are no incoming waves from infinity, the hypothetical coefficient A_3 of the incoming component is zero. Considering that the pressure and displacement field needs to satisfy the coupling conditions on the structure boundary Γ_S (at $r = R_0$) and the continuity conditions at fluid-porous media interface Γ_C (at $r = R_1$ and $r = R_2$), it follows that,

$$\frac{1}{\rho_F \omega^2} \frac{\partial p}{\partial r}(R_0) = U_0, \quad (3.6.2)$$

$$p(R_1^-) = p(R_1^+), \quad (3.6.3)$$

$$\rho_P \frac{\partial p}{\partial r}(R_1^-) = \rho_F \frac{\partial p}{\partial r}(R_1^+), \quad (3.6.4)$$

$$p(R_2^-) = p(R_2^+), \quad (3.6.5)$$

$$\rho_F \frac{\partial p}{\partial r}(R_2^-) = \rho_P \frac{\partial p}{\partial r}(R_2^+), \quad (3.6.6)$$

where U_0 is the amplitude of the radial component of the incident displacement field $\mathbf{u}^{\text{inc}}$ at $r = R_0$.

Substituting the general solution (3.6.1) in (3.6.2)-(3.6.6) and rewriting in terms of the coefficients A_1 , A_2 , B_1 , B_2 , and B_3 , a 5×5 linear system of equations is obtained and solved to compute the exact solution, both in terms of the scattered pressure and also the displacement field.

This spherical-symmetric problem has been computed by using the finite element discretization described above with a geometrical setting where $R_0 = 0.05$, $R_1 = 2R_0$, $R_2 = 2.25R_0$. The physical properties of the fluid and porous materials are given by $\rho_F = 1.213$, $c_F = 343$, $\eta_F = 1.84 \times 10^{-5}$, $\sigma = 40000$, $\phi = 0.94$, $\alpha_\infty = 1.06$, $\Lambda = 56 \times 10^{-6}$, $\Lambda' = 110 \times 10^{-6}$, $k'_0 = \eta_F/\sigma$. Since no steel wire meshes are involved in this test, its surface impedance $Z(\omega)$ will be null. The incident radial pressure wave has an amplitude $P_0 = 100$ and it is propagated at $\omega = 5000$, which leads to $U_0 \approx 4 \times 10^{-6}$.

Figure 3.6 shows the the magnitude of the real and imaginary part of the absolute error field computed between the exact solution and the finite element approximation in terms

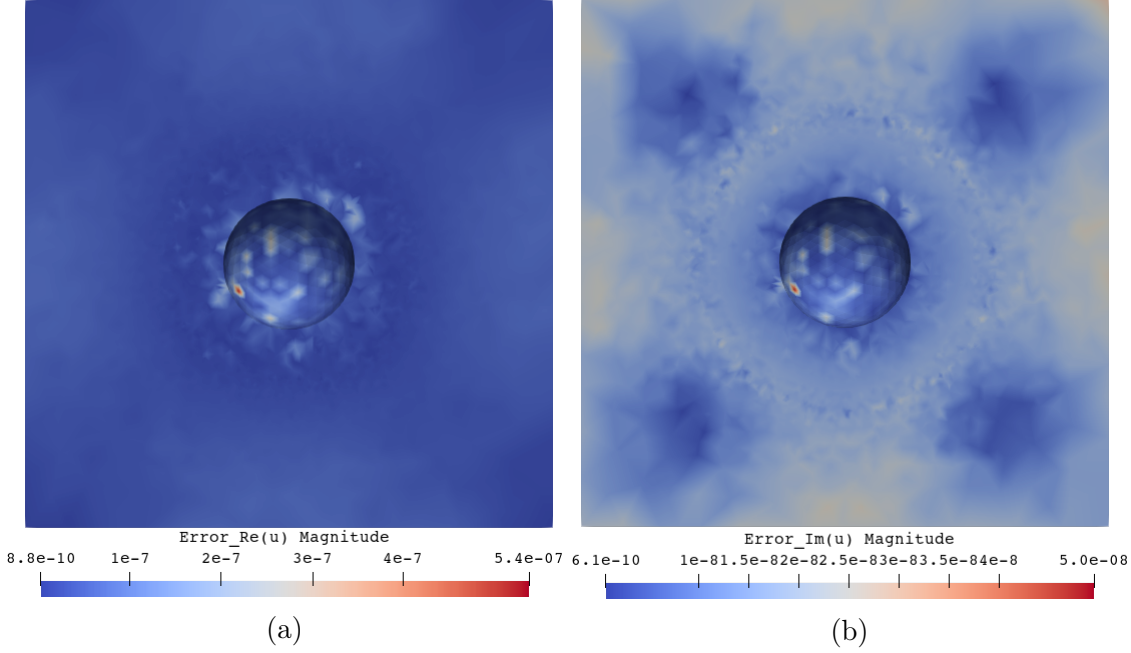


Figure 3.6: Two-dimensional vertical cut (on $x_1 = 0$) of the magnitude of the real (a) and imaginary part (b) of the absolute error computed between the exact and the finite element approximation of the displacement field for a frequency of 800Hz.

of the displacement field, this is, for each point $\mathbf{x} \in \Omega_F \cup \Omega_P$, the error field is given by

$$\begin{aligned} \text{error}_{\text{re}}(\mathbf{x}) &= ||\text{Re}(\mathbf{u}(\mathbf{x})) - \text{Re}(\mathbf{u}_h(\mathbf{x}))||, \\ \text{error}_{\text{im}}(\mathbf{x}) &= ||\text{Im}(\mathbf{u}(\mathbf{x})) - \text{Im}(\mathbf{u}_h(\mathbf{x}))||. \end{aligned}$$

As it can be observed in the plots (a) and (b) of Figure 3.6, the magnitude of the absolute errors are lower than 6×10^{-7} which leads to a pointwise maximum relative error around 15% in the real part and around 1.3% in the imaginary part. The finite element errors are mainly concentrated on the spherical boundary of the obstacle due to the poor geometrical approximation of the curve spherical boundary by means of the planar faces of the tetrahedral mesh. Notice also that despite the exact solution of the scattering problem is radial, the used mesh is unstructured and not radial symmetric.

3.7 Characterization of windscreen components

Any comparison between a mathematical model and some experimental measurements requires the use of the same geometrical setting and the identification of the physical properties of all the materials involved in the model. In this work, the physical parameters of the steel wire mesh and the porous materials are analyzed in detail.

3.7.1 Insertion Loss measurements

The insertion loss experiments were conducted at the anechoic chamber of Microflow Technologies using a PU regular probe. A simple experimental investigation was performed to quantify the insertion loss of two different steel meshes and porous windscreens. The probe was mounted vertically in an anechoic chamber against a sound source facing perpendicular to the direction of the probe pillars (most sensitive direction). A sound signal was measured for 30 seconds and the associated pressure and particle velocity spectra were then computed. Subsequently, porous windscreens were introduced over the probe and the experiments were repeated. Multiple trials were conducted to ensure the consistency of the data and minimize the re-positioning bias. Figure 3.7 shows the pressure and particle velocity spectra measured using the probe covered with a single layer foam windscreen.

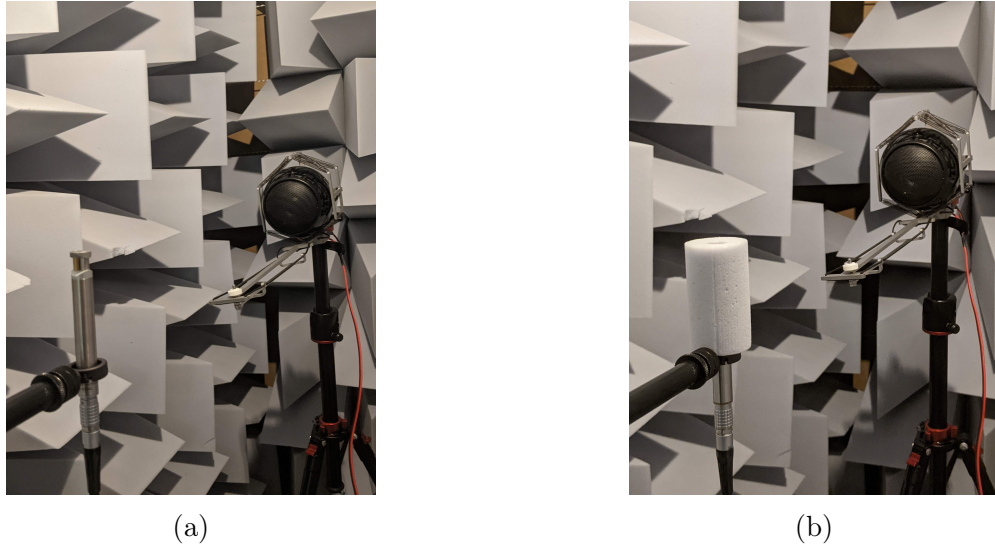


Figure 3.7: Measuring acoustic signals with and without windscreen

The porous foam windscreens were made with a cylindrical shape using two materials - EKI Melamine and EKI Filter Foam 90 PPI. Several melamine windscreens and filter foam windscreens were fabricated with the intention to eliminate the biases towards materials and geometrical parameters. Figure 3.8 shows the two different cylindrical porous windscreens that were used in the study.

Experimental results are presented in Figure 3.10. As shown, the evaluated windscreens have a significant influence over high frequencies (>5 kHz), with high repeatability between different trials. Similar behavior is observed for both transducers, but in the case of the particle velocity, the lower frequency range (<300 Hz) is also biased by the porous layer surrounding the probe. Frequencies below 100 Hz were omitted due to the limitations of the testing environment (cut-off frequency of 200 Hz). From these results, it can be inferred that the selection of windscreen materials has a significant impact on the sensor performance, especially for particle velocity transducers, and therefore an efficient windscreen design becomes critical for outdoor applications.



Figure 3.8

Figure 3.9: Two different porous windscreens used in this investigation.

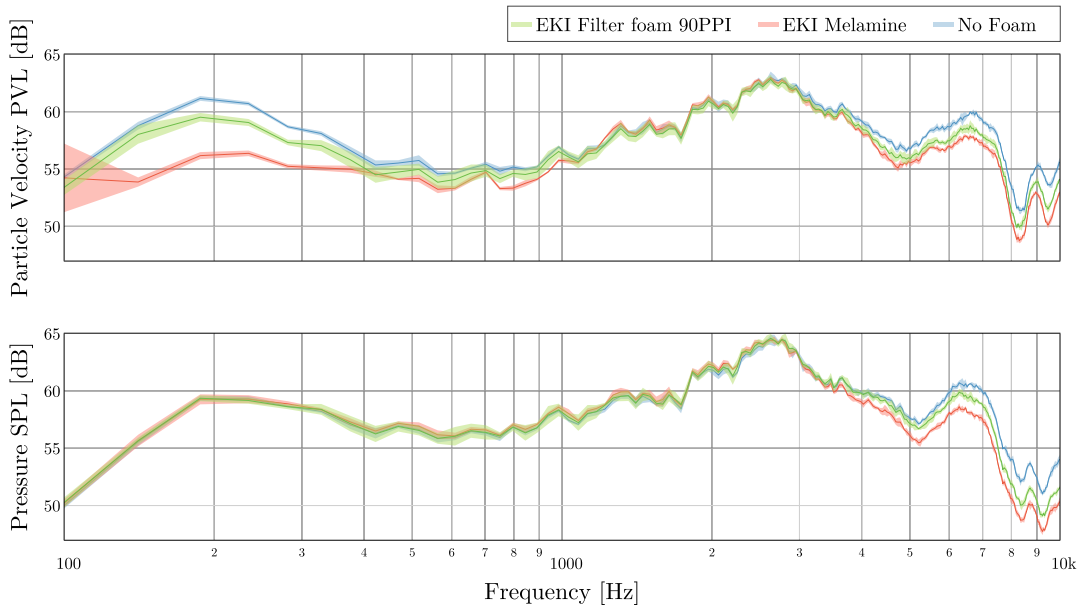


Figure 3.10: Experimental measurement of particle velocity and pressure with and without porous layer.

3.7.2 JCAL parameter estimation

Before validating the model against experimental results, it is necessary to obtain the parameters that define the properties of the evaluated materials. The parameters of the steel meshes are simply the pore size, area and thickness of the mesh and are easily obtained from the manufacturer. However, the six JCAL parameters that describe the porous material

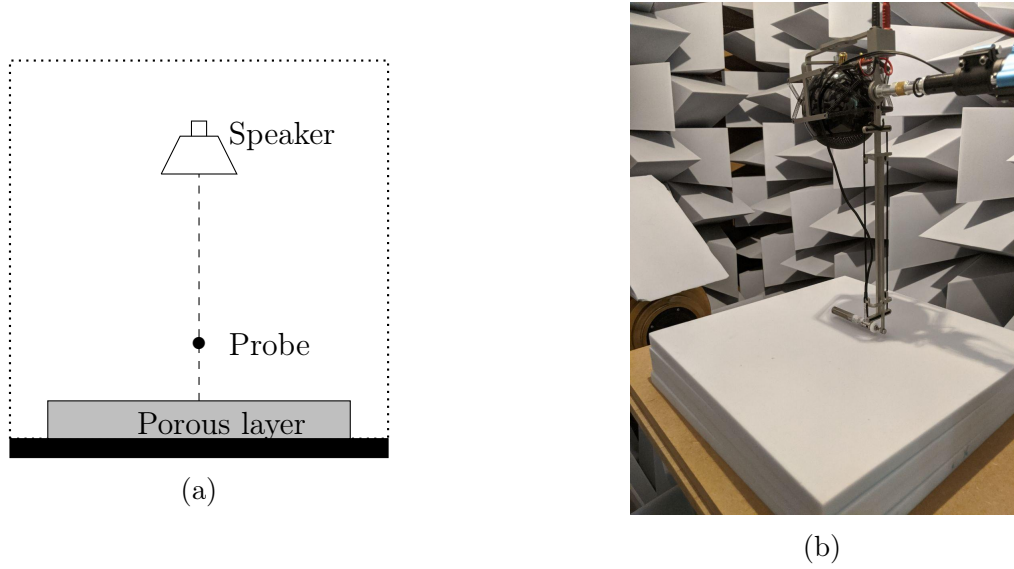


Figure 3.11: The schematic of the Microflow in situ setup (a) and the apparatus (b) installed in an anechoic chamber.

(see Section 3.2.2) are volumetric quantities which are not easily found. While there exist experiments to measure ϕ and σ , there are no standard methodologies for measuring α_∞ , Λ , Λ' and k'_0 . The standard methodologies to measure the acoustical performance of porous materials often rely on surface absorption and impedance measurements of the sample conducted either in a Kundt's tube (ISO 10534 and ASTM E1050) or a reverberant room (ISO 354 and ASTM C423) under laboratory conditions. On the other hand, there are also non-standardized in-situ methodologies, such as the P-U “in-situ” method, which have demonstrated to yield similar results [27]. The setup consists of a pressure-particle velocity (PU) probe fixed at a specific distance away from a spherical sound source. The source is attached to a metallic frame with a suspension system composed by multiple springs, isolating structural vibrations from the source to the measuring sensor. The setup is placed in close proximity to the material evaluated along the normal direction as shown in the schematic shown in Figure 3.11a. While the setup can be used in existing measuring environments, it was placed in an anechoic chamber to eliminate any influence that the surroundings may pose (especially at lower frequencies). The sample is ensured to be flat and made of the same material used in the sensor windscreens. This is necessary to avoid any impact of an oblique angle of incidence on the sample surface and also the finite-sample effect [27], since the probe coverings are relatively small. The sample was placed against a rigid baffle reflecting all the incident acoustic energy. Figure 3.11(b) shows the measuring apparatus in an anechoic chamber.

The experimental reflection coefficient R_{exp} of the porous sample was measured using the acoustic field method, assuming that the field is comprised solely of plane waves. The probe measures the acoustic pressure and the particle velocity at its position and the reflection coefficient can be computed *a posteriori* using the ratio of the direct acoustic signal from

the speaker and the reflected field from the impedance plane. The experimental surface impedance (Z_{exp}) of the porous material may then be computed based on the measured reflection coefficient R_{exp} using a plane-wave assumption as,

$$Z_{\text{exp}} = \frac{1 + R_{\text{exp}}}{1 - R_{\text{exp}}}. \quad (3.7.1)$$

An inverse model is subsequently formulated by evaluating the measured impedance values against the porous parametric model of surface impedance. The parametric impedance $Z(\omega; \mathbf{p})$ is modeled using a plane wave analysis [31] as,

$$Z(\omega; \mathbf{p}) = \sqrt{\phi^2 \rho_P(\omega; \mathbf{p}) K_P(\omega; \mathbf{p})} \coth \left(i \frac{\omega^2 \rho_P(\omega; \mathbf{p})}{K_P(\omega; \mathbf{p})} l \right), \quad (3.7.2)$$

with l as the thickness of the sample and the JCAL parameter vector, $\mathbf{p} = (\phi, \sigma, \alpha_\infty, \Lambda, \Lambda', k'_0)^T$. The parameters is estimated by fitting against the experimental impedance values $Z_{\text{exp},j}(\omega_i)$ measured for $j = 1, \dots, N$ trials in a sequence of M angular frequencies ω_i , with $i = 1, \dots, M$, which are logarithmic equispaced. Hence, the JCAL parameters are given by

$$\mathbf{p} = \underset{\mathbf{p} \in (\mathbb{R}^+)^6}{\text{argmin}} \sum_{j=1}^N \left(\frac{\sum_{i=1}^M |Z_{\text{exp},j}(\omega_i) - Z(\omega_i; \mathbf{p})|^2}{\sum_{i=1}^M |Z_{\text{exp},j}(\omega_i)|^2} \right). \quad (3.7.3)$$

The impedance measurements were conducted at the sensor calibration chamber at Microflown Technologies. Numerous flat samples of the EKI Melamine and EKI Filter Foam material were obtained from the manufacturer of dimension $50 \times 60 \times 2 \text{ cm}^3$. Since the sample thickness were relatively thin, an individual layer was observed to barely exhibit any absorption and reflected most incident sound energy. To quantify the absorptive properties definitively, multiple layers were placed upon each other as is shown in Figure 3.11(b).

Figure 3.12 shows the JCAL parametric fitting performed on the experimental results obtained on the foams. The model considered the multiple layers of the material as a homogenous block of total thickness. The range of the frequencies varied for the materials to ensure a better fit and to avoid non-physical parameter values. The fit shows a qualitative match with the experimental results with a better agreement at frequencies over 1000Hz. The parameters were obtained by fitting the complex impedance values using a bounded truncated Newton algorithm and Table 3.1 highlights these estimated parameters. It is noted that the six JCAL parameters over-parameterize the surface impedance measurements, and the porosity, ϕ was fixed during the optimization. Moreover, it should be mentioned that the inverse optimization with the JCAL model is extremely sensitive to constraints and bounds and yields atypical non-physical values. An improved optimization algorithm with a lower parametrization or remodeled cost function could yield better results.

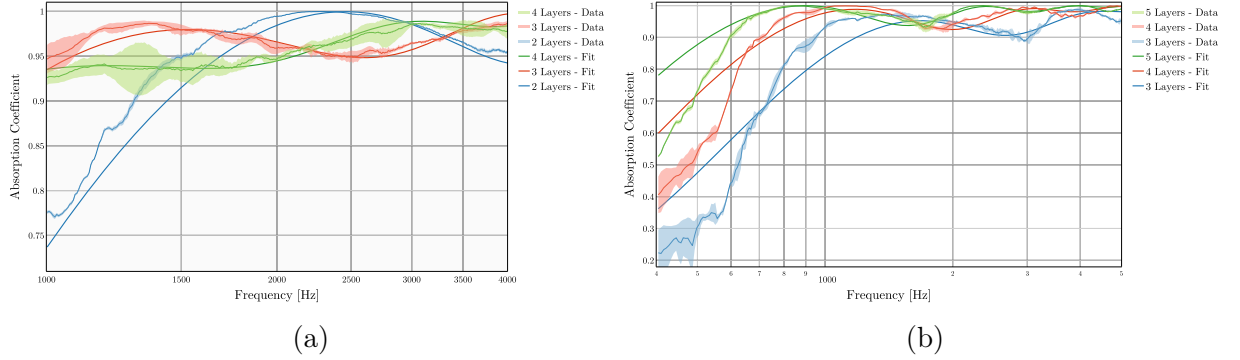


Figure 3.12: Absorption measurements for different layers of EKI Melamine (left) and the EKI Filter 90PPI foam (b) with parametric fitting

Material	ϕ	α_∞	σ	Λ	Λ'	k'_0
EKI Melamine	0.95	1.1	18443	1.84×10^{-4}	4.07×10^{-4}	1.49×10^{-10}
EKI Foam	0.95	1.1	6956	2.58×10^{-4}	1.36×10^{-4}	3.48×10^{-10}

Table 3.1: Fitted JCAL properties to impedance data of the foams

3.8 Simulation Results and Discussion

3.8.1 Package Gain

Sound intensity PU probes contain sensitive elements that are often wrapped with a protective mesh for increasing their mechanical robustness. As a result, this additional element can influence the acoustic field in the vicinity of the transducer. *Package gain* of an acoustic sensor refers to the impact on the acoustic signal measured by the sensor while introducing packaging around it. A sketch of the problem is illustrated in Figure 3.13.

Sensor manufacturers carefully consider the design of the probe packaging around the transducer and it undergoes a lot of iterations during the design process. The prototypes are accounted for their impact on the the acoustic particle velocity signal or pressure signals and the sensors and therefore are calibrated accordingly. The Microflow PU probe evaluated in this study especially shows a considerable package gain in particle velocity due to the channeling effect of the probe pillars around the sensor position. The company reports a gain of more than 10 dB at the sensor position compared to a single transducer without surrounding elements. In this section, we utilize the mathematical model developed in Section 3.2 to predict the package gain of a Microflow PU probe housing around an ideal sensor.

The geometric specifications of the probe were measured and introduced into a computer-aided design (CAD) model using SALOME. A 3D Cartesian computational domain of $10.674 \times 10.674 \times 12.453 \text{ cm}^3$ was made to represent the acoustic fluid around the probe.

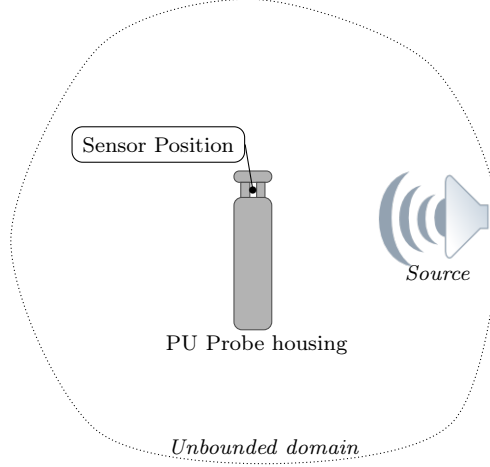


Figure 3.13: Schematic of the package gain problem

The domain was enveloped by a Cartesian perfectly matched layer of 4.4475 cm thickness and a mesh is generated with 148906 vertices and 838058 cells using the open-source software Gmsh. Of particular note is the conformality of the mesh to interfaces of the fluid and the PML to ensure the interface conditions between the regions with different governing equations. Particularly, since the PML has a piece-wise definition of the equations along the x_1 , x_2 , x_3 directions - the mesh is made accordingly conformal by splitting into 26 subdivisions. A plane wave of unit amplitude is assumed to be incident on the PU probe in a direction perpendicular to that of the pillars. Figure 3.14 shows the contours of the real and imaginary parts of the resulting scattered displacement field. The component displayed is along the direction of the plane wave for a frequency of 1976Hz.

The results indicate an increase in the real part of particle displacement between the probe pillars. This corresponds to the physical attributes of the channeling effect. Also, the scattered field is absorbed entirely at the PML boundaries ensuring a complete absorption of the outgoing waves.

In addition, the total displacement field was also calculated by adding the incident plane wave and the scattered field. In this case, the increase in the scattered acoustic field also corresponds to an increase of the total displacement field between the probe pillars as indicated in Figure 3.15. Also visible are the diffraction of the incident plane wave around the probe corners.

On the other hand, a frequency response of the probe package gain was also calculated by changing ω in the model. Since the linear system is frequency dependent, the matrices have to be re-assembled and solved. The total displacement field can be readily used to compute the particle velocity $\mathbf{v}_F$, since $\mathbf{v}_F = -i\omega\mathbf{u}_F$. The pressure fields can also be computed by performing a L^2 -projection (as discussed in Chapter 2). The sensor gain was then computed as the ratio between the particle velocity and pressure signals with respect to the incident plane wave field. Figure 3.16a provides the estimated frequency response introduced by the package gain for the Microflow PU probe evaluated design. The results

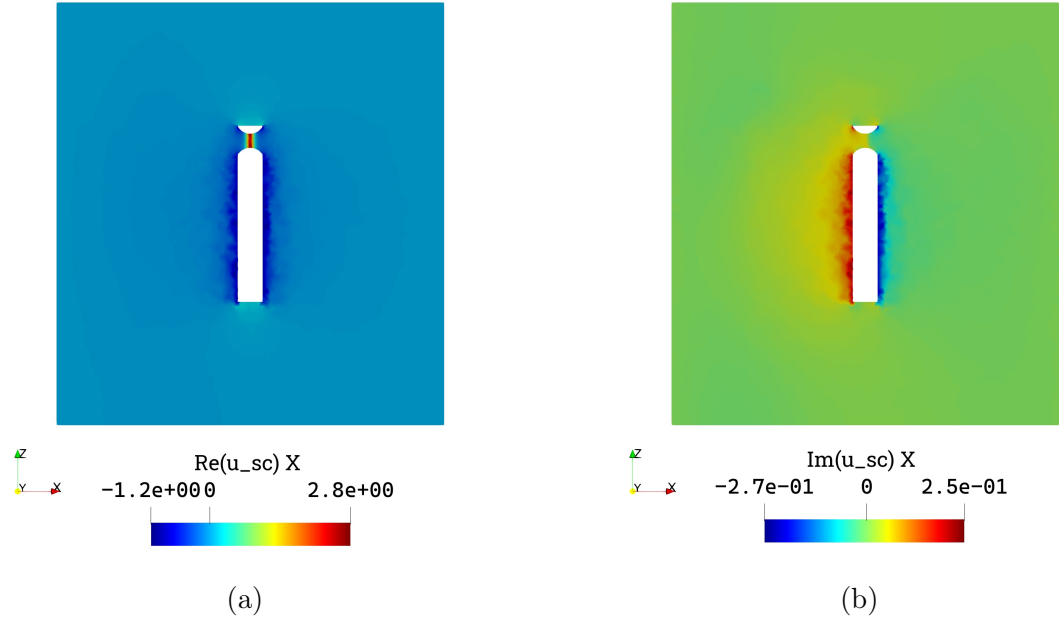


Figure 3.14: The real (left) and imaginary part(right) of the x_1 -component of the scattered displacement field including the PML domain excited at 1976 Hz.

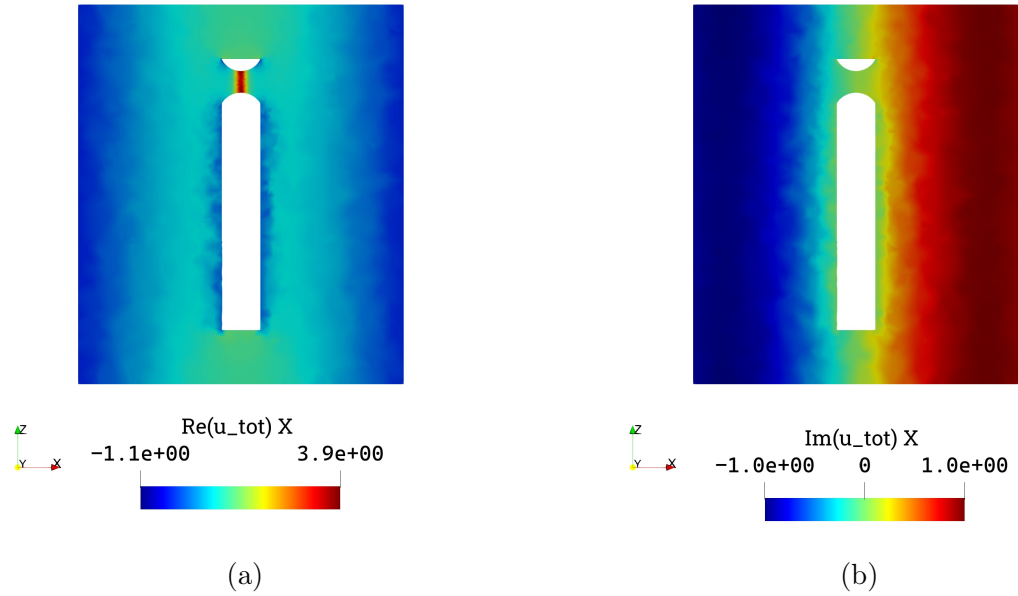


Figure 3.15: The real part(left) and the imaginary part(right) of the x_1 -component of the total displacement field (probe only) at 1976 Hz frequency

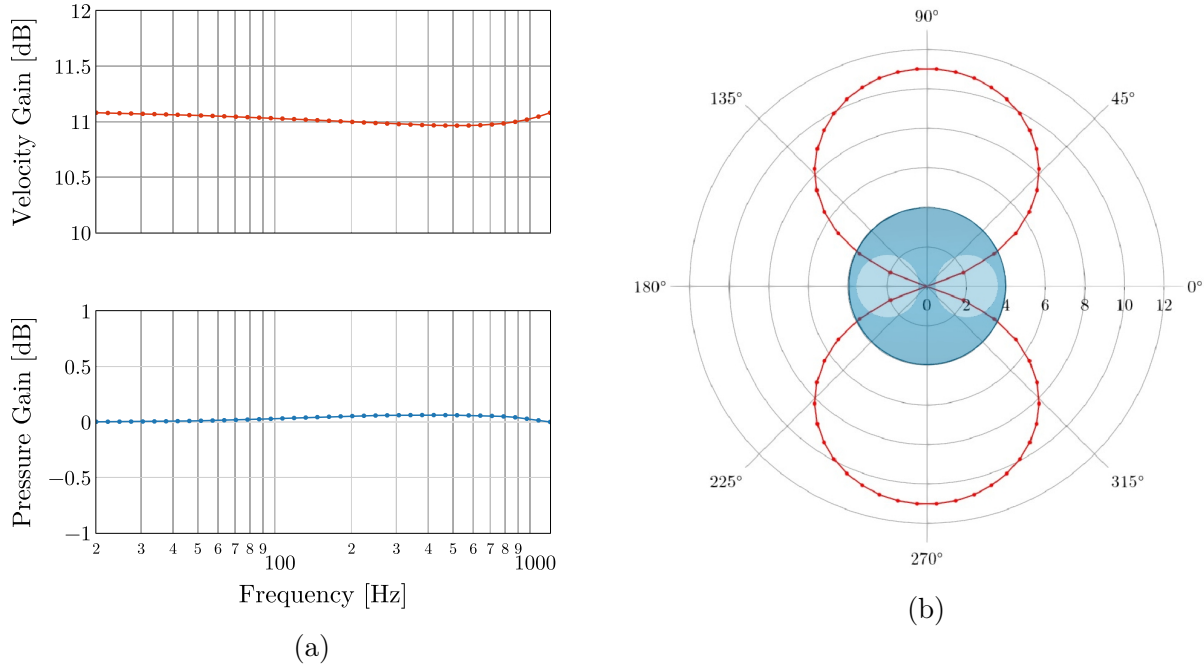


Figure 3.16: Frequency response of the package gain of the PU regular probe (left) and the polar directivity pattern of the particle velocity gain for 100Hz frequency (right), shown with respect to the orientation of the probe pillars.

clearly indicate a 11dB particle velocity gain, well inclined with the estimates in the design reports, while the pressure gain is negligible.

Finally, the direction of the incident plane wave with respect to the probe was changed to compute the directional influence of the probe housing. Figure 3.16b demonstrates that the ideal figure-of-eight directivity pattern is achieved for a frequency of 100Hz, confirming that the probe design does not have a negative impact on the directional response.

3.8.2 Steel Mesh

To acoustically characterize the steel mesh on the periphery of the probe boundary shown in Figure 3.17, the key element is to determine the complex impedance coefficient *a priori*.

Following the simple model proposed by Maa [56], the parameters to determine this coefficient are: the thickness of the panel t , the equivalent diameter or the pore size d and the perforation ratio p of the interface. The thickness of the panel and the pore sizes are readily available through the manufacturer, which are shown in Table 3.2.

To compute the acoustic signal gain numerically, it is required that the mesh is conformal at the steel mesh interface so as to impose the interface conditions along it. A plane wave

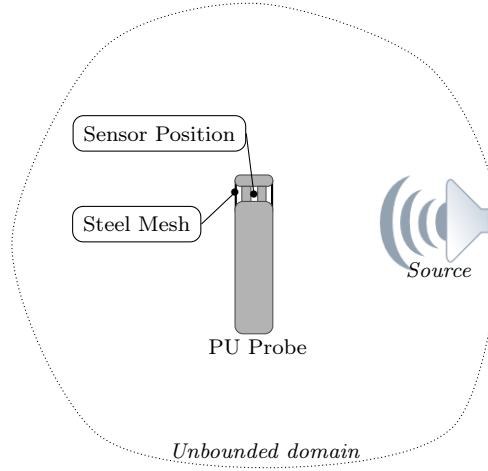


Figure 3.17: Schematic of the steel mesh problem

Mesh type	d	t	p
Steel grid	50×10^{-6}	77×10^{-6}	0.55

Table 3.2: Geometrical properties of the steel wire mesh (d : diameter of the pores, t : thickness of the mesh, p : perforation ratio).

of unit magnitude is assumed incident on the the PU Probe in a direction perpendicular to the pillars. For the chosen frequency, the FEM matrices were assembled using FEniCS. To solve the linear system, the MUMPS solver was used. Figure 3.18 shows the plots of the scattered displacement field for a frequency of 1200Hz on a plane slice perpendicular to the pillars.

The real part of the displacement field between the pillars are still increased due to the narrow channel and are largely unaffected by the presence of the steel mesh. However, in this case the imaginary part is visibly affected at the steel mesh.

The total displacement field was obtained by adding the incident plane wave part to the scattering solution. Figure 3.19 shows the total field of displacement of the acoustic signal. As shown, the field retains the increase of the local displacement between the probe pillars whereas the impact of to the steel mesh is negligible.

Subsequently, the particle velocity field and the pressure field were computed as described in Section 3.8.1 for a set of frequencies. The fields are sampled and compared against the similar values obtained from the package gain study. Figure 3.20 shows the effective gain computed and experimentally observed due to the introduction of a steel mesh between 40 Hz and 800 Hz. The numerical results display a qualitative agreement with the obtained data. The steel mesh displays a higher particle velocity gain - up to 6 dB at lower frequencies. However, the pressure gain is negligible and this emphasizes the different characteristics between the sound pressure and particle velocity transducers.

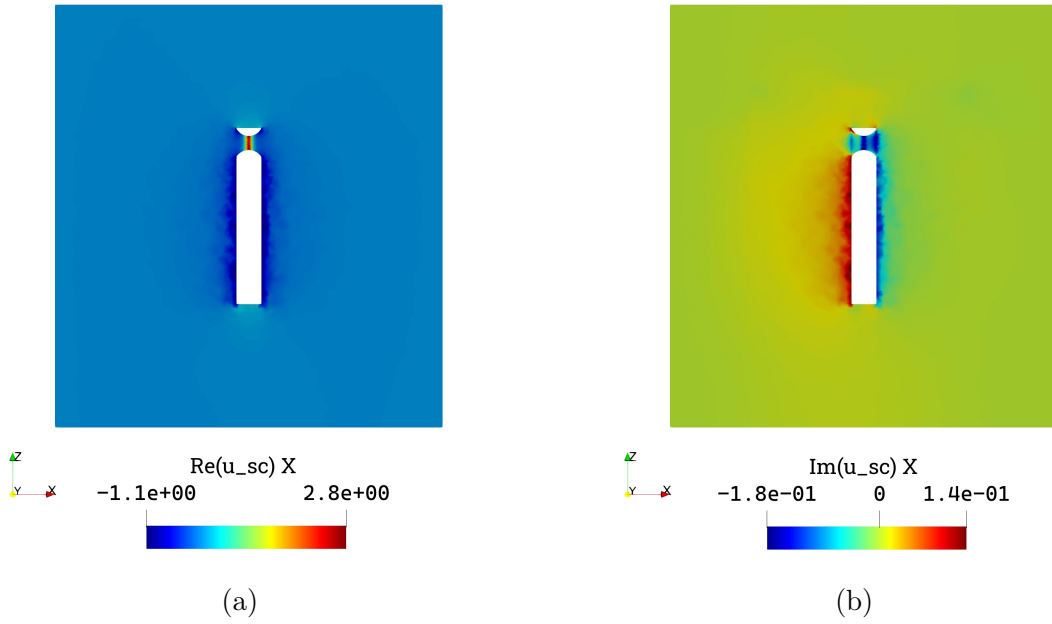


Figure 3.18: The real part(left) and the imaginary part(right) of the X-component of the scattered displacement field including the PML domain at 1200 Hz frequency for the PU regular probe covered in steel mesh.

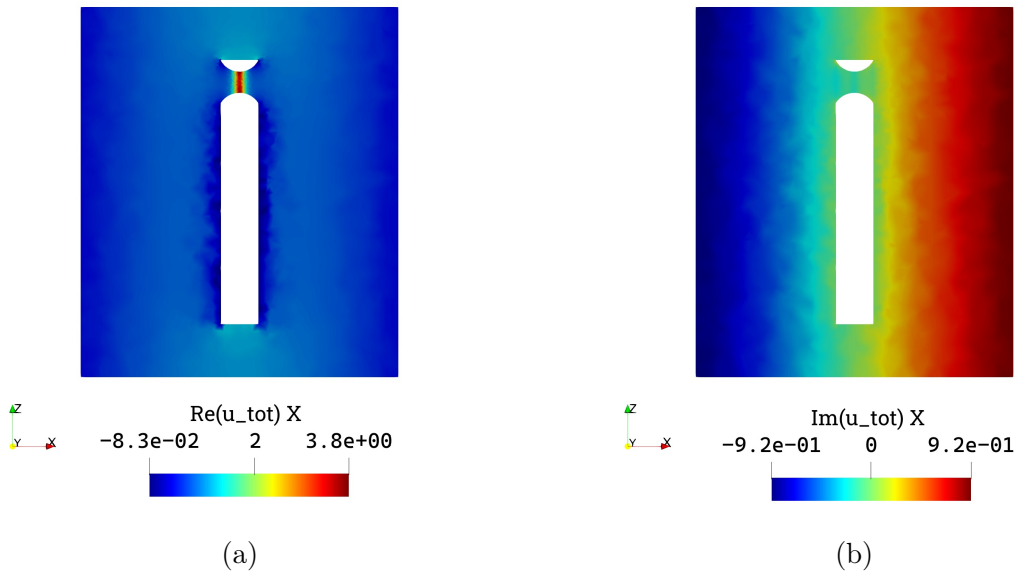


Figure 3.19: The real part(left) and the imaginary part(right) of the X-component of the total displacement field at 1200 Hz frequency for the PU regular probe covered with steel mesh.

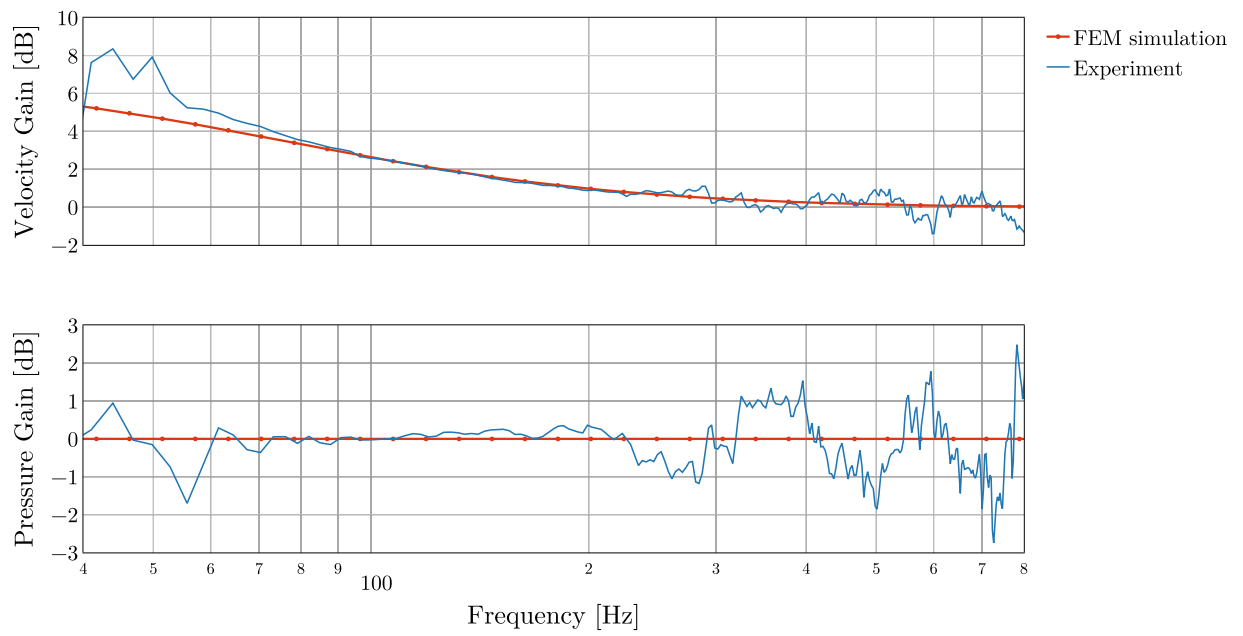


Figure 3.20: Frequency response of the particle velocity and the pressure gain due to a steel mesh

3.8.3 Insertion Loss of Porous Windcreens

A sketch of the problem evaluated in this section is displayed in Figure 3.21, accounting for the probe geometry and the cylindrical porous windscreen.

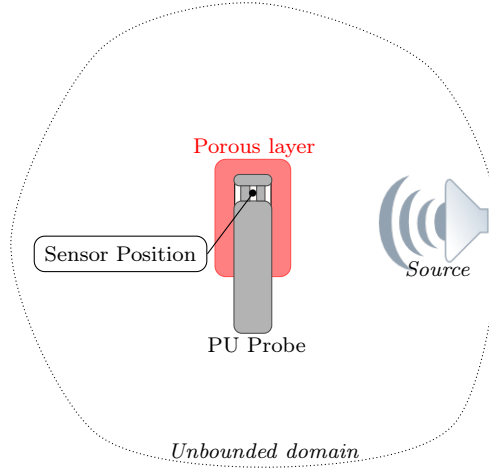


Figure 3.21: Schematic of the insertion loss measurement problem

Cylindrical porous layers display a considerable dissipative effect which are numerically characterized using the six JCAL parameters for the material. These parameters are obtained from the inverse model using the in situ measurement data on flat samples of the same material. To ensure that the integral formulation of the porous governing equations are consistent with the subdomain while assembling the finite element matrices, the meshes are again required to be conformal with the porous layer boundaries. A mesh is hence generated using Gmsh with the same geometrical parameters as the earlier section, and a scattering problem is formulated to measure the influence of the porous layer around the probe.

An incident plane wave of unit magnitude is assumed perpendicular to the direction of the pillars. The subdomains are clearly demarcated to ensure the assembling of the FEM matrices in the respective regions. The complex-valued dynamic porous density and the bulk modulus are computed for each frequency based on the parameters gotten from inverse optimization procedure. Since the plane wave is not a solution to the porous governing equation, care is taken to handle the contribution from the incident plane wave terms on the linear form of the variational equation. The Dirichlet boundary conditions are imposed on the probe boundary and on the PML outer boundary, solving the linear system using the MUMPS linear algebra solver. Figure 3.22 shows the scattered displacement field in the entire computational domain.

It is evident that the porous layer exhibits a significant influence on the acoustic fluid, since their outline is easily discernible from the contours. It should be noted the influence on the imaginary part of the displacement field and a significant rise in the displacement magnitude between the pillars.

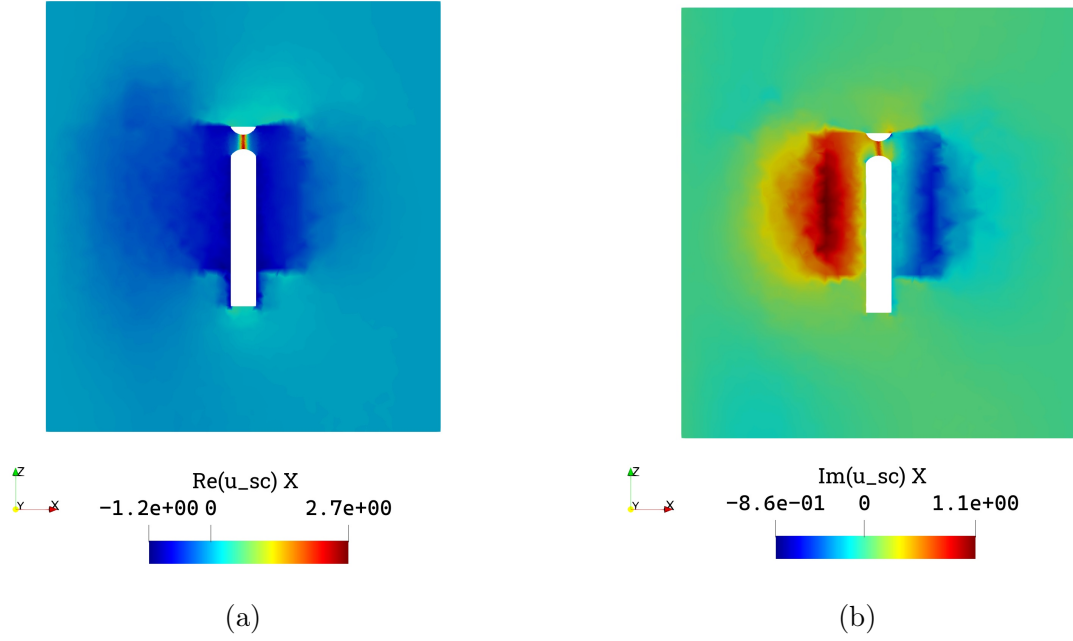


Figure 3.22: The real part(left) and the imaginary part(right) of the X-component of the scattered displacement field including the PML domain at 1976 Hz frequency for probe covered with the EKI Melamine windscreen.

The total displacement field (Figure 3.23 and 3.25) echoes similar observations from the scattered field. The PML domain is omitted in these figure for its lack of sensible information. The results emphasize the porous dissipative effects on the flow and a comparison of the plots indicate that the EKI Filter Foam 90PPI windscreen causes a slightly lower impact to the total acoustic field than the EKI Melamine foam for the same thickness. This is in accordance to the experimental results obtained in section 3.7.1.

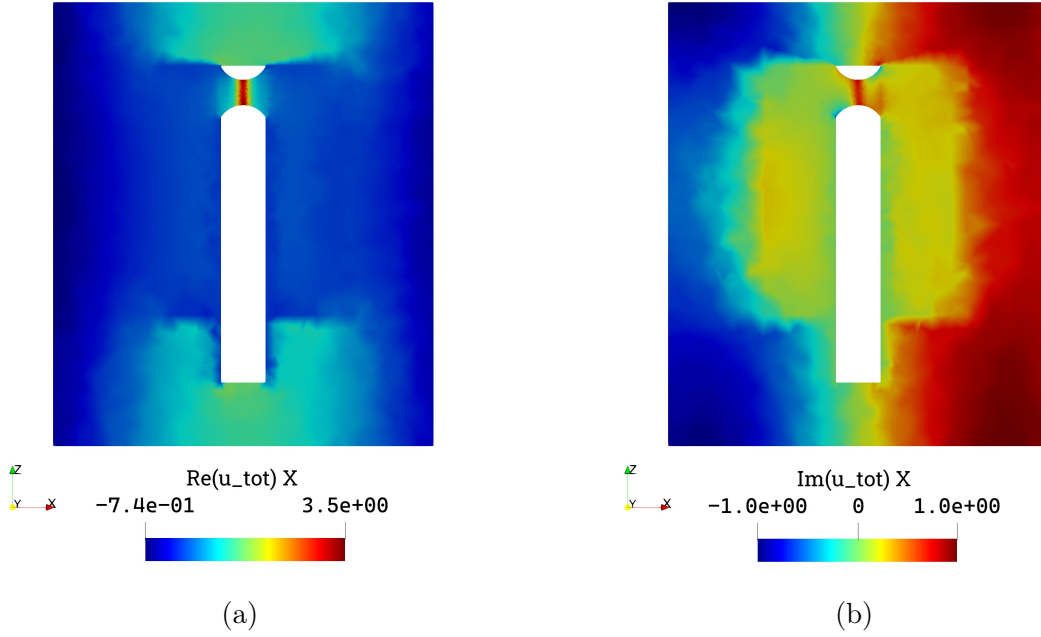


Figure 3.23: The real part(left) and the imaginary part(right) of the X-component of the total displacement field at 1976 Hz frequency for probe covered with the EKI melamine windscreen.

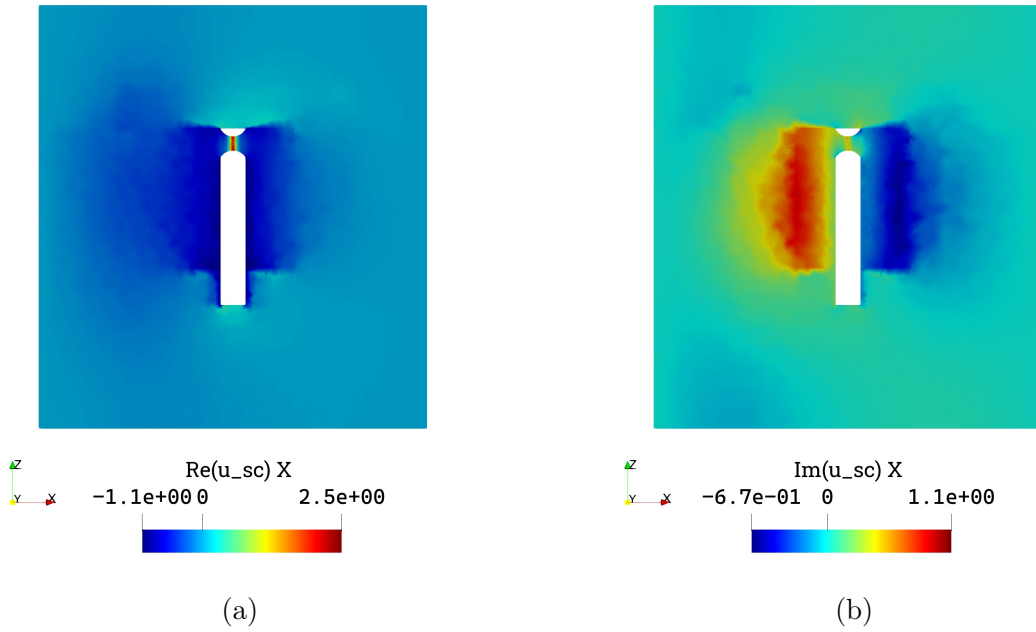


Figure 3.24: The real part(left) and the imaginary part(right) of the X-component of the scattered displacement field including the PML domain at 1976 Hz frequency for probe covered with EKI90 windscreen only.

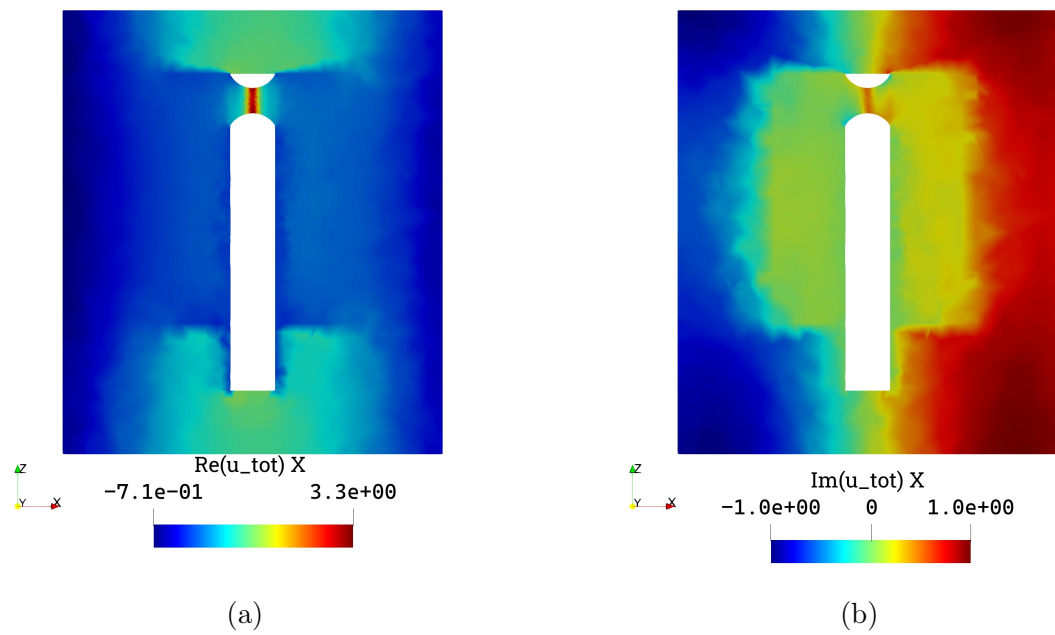


Figure 3.25: The real part(left) and the imaginary part(right) of the X-component of the total displacement field at 1976 Hz frequency for probe with EKI90 windscreen only.

3.8.4 Coupled impact of Steel mesh and Porous Windscreens

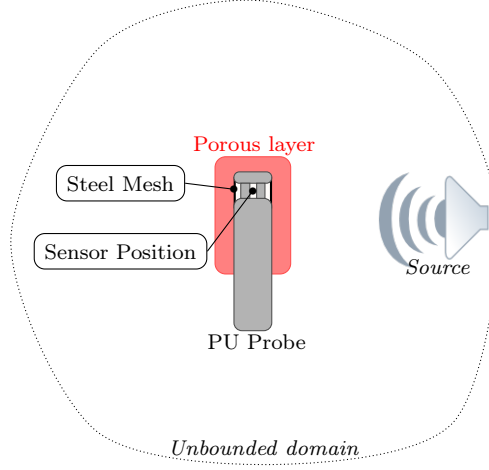


Figure 3.26: Schematic of the coupled problem of PU probe covered in steel mesh and porous windscreen

The final results portray the finite element formulation of the entire ensemble of acoustic media - the PU probe housing, the steel mesh and the porous layer. All of the simulations have been implemented for a number of frequencies between 20-3000Hz. We report here the results here with the different configurations of steel meshes and porous layers to illustrate their differences in the acoustic flow.

Two steel meshes with the parameters 3.2 and two porous layers - the EKI Melamine and EKI Filter Foam 90PPI with the JCAL parameters obtained using the inverse characterization were utilized for the numerical simulation. Since the porous layers were fabricated with the same physical dimensions, the exact same geometry and mesh were used in all case albeit with the different physical properties. A computational domain of $10.674 \times 10.674 \times 12.453 \text{ cm}^3$ was considered around the center of the probe. A Cartesian perfectly matched layer (PML) of uniform thickness with 4.4475 cm was included across the domain to replicate the non-reflecting boundary condition. The mesh was constructed using with particular attention to being conformal across all subdomains to ensure numerical accuracy during assembly. Further, the mesh was locally refined in the probe pillar region and the probe body to capture the curved geometries accurately.

An plane wave of unit magnitude was considered impinging on the probe on the direction perpendicular to the probe pillars (the most sensitive orientation of the probe). While the parameters of the steel mesh were used as obtained from the manufacturer, the porosity of both the porous layers were afixed at 0.95, while obtaining the other parameters by fitting them to the experimental surface impedance data. The scattered displacement field is displayed for all the configurations which includes the PML domain - and it visibly illustrates the absorbing behaviour as you approach the domain boundary. The plots also clearly display the influence of all the acoustic media on the plane wave, particularly the

diffraction and any internal reflection and dissipative behavior. The finite element approximation is done using Raviart-Thomas basis elements which ease up the specification of boundary conditions across the conformal boundaries and interfaces since the basis functions are by definition normal to the faces of the mesh. The finite element matrices are assembled and subsequently solved using the MUMPS linear algebra solver.

In all of the results, the component total displacement fields along the direction of the plane wave is shown since it displays the scattering vividly. The slices are displayed for each configuration, enabling a visualization of the acoustic field within the porous layer as well as within the steel mesh - of special relevance in the industry. The porous layer outlines are distinctly visible in the reported plots, highlighting again, the importance of characterization of their behavior towards predicting the acoustic field. Comparison of the plots can crucially highlight the physical behavior of the porous layers. Comparing between Figures 3.27 and 3.29, it can be seen that while the EKI Melamine layer amplifies the incident plane wave more than the EKI Filter foam 90PPI, it also displays the maximum dissipative behavior. As expected, the steel meshes virtually do not extensively influence the total acoustic field and can be considered almost transparent for the reported frequency.

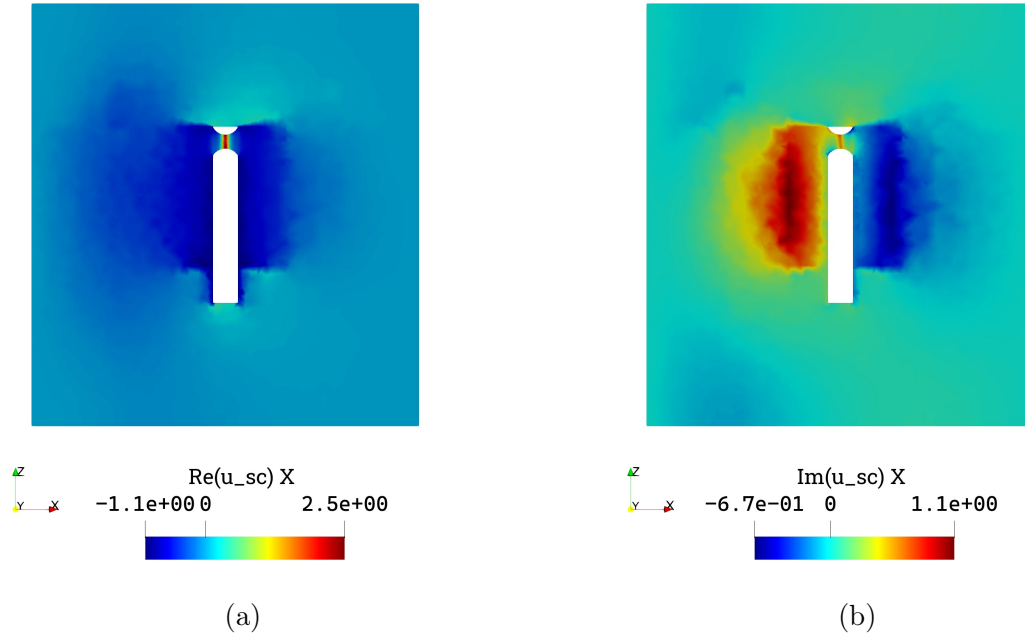


Figure 3.27: The real part(left) and the imaginary part(right) of the X-component of the scattered displacement field including the PML domain at 2014 Hz frequency for the PU probe with the EKI Melamine windscreen and steel mesh layers.

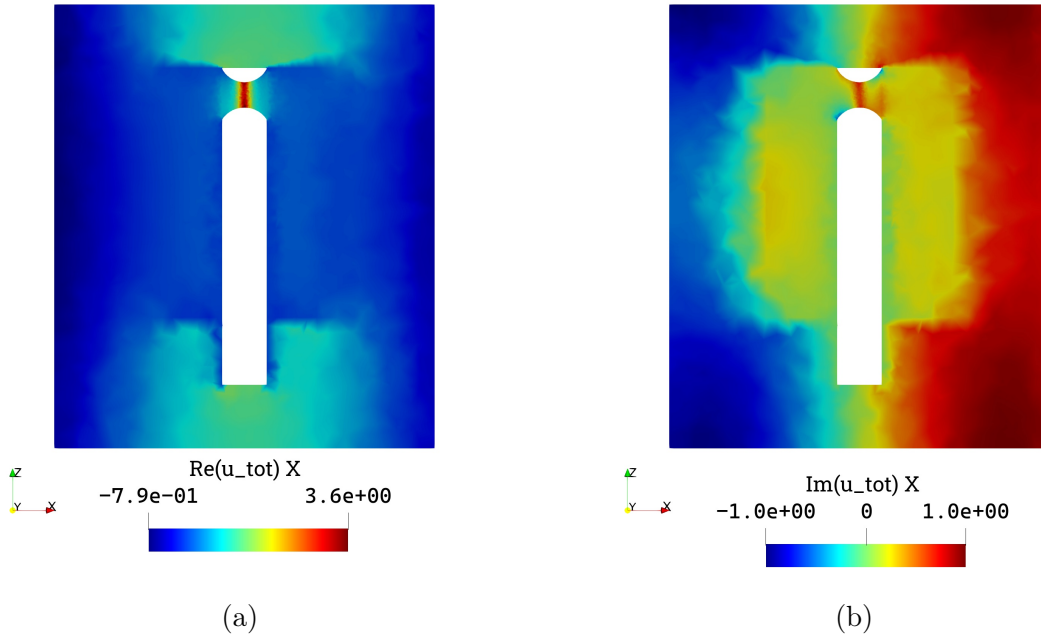


Figure 3.28: The real part(left) and the imaginary part(right) of the X-component of the total displacement field at 2014 Hz frequency for probe with EKI Melamine windscreen and steel mesh layers.

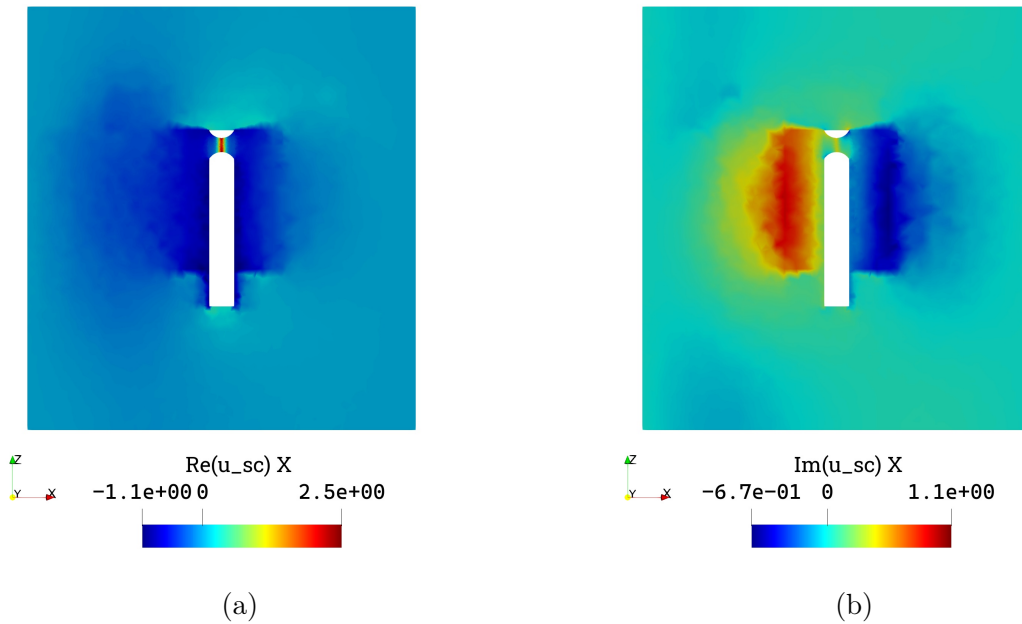


Figure 3.29: The real part(left) and the imaginary part(right) of the X-component of the scattered displacement field including the PML domain at 1976 Hz frequency for the PU probe with EKI Filter Foam 90PPI windscreen and steel mesh layers.

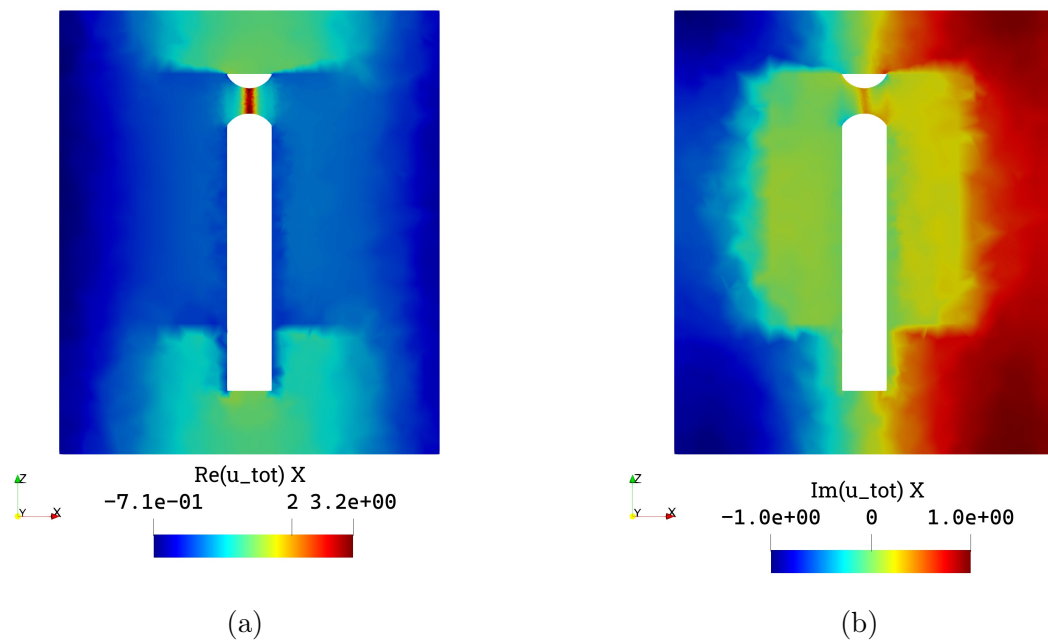


Figure 3.30: The real part(left) and the imaginary part(right) of the X-component of the total displacement field at 1976 Hz frequency for the PU probe with EKI90 windscreen and steel mesh layers.

Chapter 4

A Darcy-Forchheimer-Galbrun linear model for small perturbations of a flow in rigid-frame porous materials

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Keywords: Vibrations. Poro-aeroacoustics. Galbrun's equations. Darcy-Forchheimer porous model.

4.1 Introduction

The linear governing equations for small isothermal perturbations of a porous material is deduced in the present manuscript, assuming that reference state includes the presence of an underlying flow. The heuristic derivation of this model is made formally starting from the classical hypothesis of continuum mechanics, that is, the mass and momentum conservation equations, which have been modified (according to the heuristic model proposed in [70]) to take into account the typical dissipative effects of a porous material and the microscopic porosity of the porous frame and assuming that small perturbations keep the macroscopic temperature of the material points in the porous material. The non-linear conservation laws are written classically in terms of the material velocity field described in Eulerian coordinates, which involves not only the small perturbation Lagrangian displacement field but also the velocity associated with the underlying flow (which will be assumed known *a priori*). Then those equations are linearized by computing the derivatives of the corresponding response functions. Consequently, the proposed model consists a general version of governing equations of isothermal dissipative vibrations from a flow, including body forces. Finally, some simplifications are obtained, in the the case where the porous material is assumed as a compressible fluid, and in the case where the porous material is initially at rest (no flow). The deduced model, called as Darcy-Forchheimer-Galbrun model describes not only acoustic waves but also internal gravity waves. The proposed heuristic derivation of the Darcy-Forchheimer-Galbrun model for poro-aeroacoustic wave propagation follows the arguments used classically to derive the Galbrun's model (see [18] for more details).

4.2 Continuum mechanics notation

Firstly, some definitions are necessary from the basic notions and results of continuum mechanics, which will be used throughout this entire chapter. For this purpose, Gurtin notation will be followed (see for instance [46] or [19]). Notice that the notation used in this chapter is not consistent with the rest of the chapters of this dissertation thesis, mainly due to the fact that it is necessary to denote and distinguish clearly material and spatial points, the pressure fields, etc.

4.2.1 Lagrangian and Eulerian description of a motion

The three-dimensional affine space (consists in points, vectors and an addition operation between points and vectors) is denoted by $\mathcal{E}$ and it will be based on the standard Euclidean linear space of vectors $\mathcal{V}$ endowed with the Euclidean distance as a usual metric space.

Definition 4.2.1. A body $\mathcal{B}$ is a region of the Euclidean space $\mathcal{E}$ with regular (smooth) boundary. Body $\mathcal{B}$ will be used as the reference configuration (steady-state or initial position of a given body). Those spatial positions in $\mathcal{B}$ are called material points.

Definition 4.2.2. A deformation of $\mathcal{B}$ is a smooth one-to-one mapping $\mathbf{f}$ which maps $\mathcal{B}$ onto a closed region in $\mathcal{E}$ and it holds

$$\det \operatorname{grad} \mathbf{f} > 0$$

Definition 4.2.3. A motion of $\mathcal{B}$ is a smooth function, belonging to $C^3(\mathcal{B} \times \mathbb{R})$,

$$\mathbf{X} : \mathcal{B} \times \mathbb{R} \rightarrow \mathcal{E},$$

such that, for each fixed t , the mapping $\mathbf{X}(\cdot, t)$ is a deformation of $\mathcal{B}$.

The spatial point x where the material point $p \in \mathcal{B}$ is placed at time t is given by to $x = \mathbf{X}(p, t)$. Hence, the region $\mathcal{B}_t^{\mathbf{X}}$ of the affine space occupied by the body $\mathcal{B}$ at time t following the motion $\mathbf{X}$ is defined by

$$\mathcal{B}_t^{\mathbf{X}} = \mathbf{X}(\mathcal{B}, t).$$

Definition 4.2.4. The trajectory of body $\mathcal{B}$ by the motion $\mathbf{X}$ is the set of the affine space

$$\mathcal{T}_{\mathbf{X}} = \{(x, t) : x \in \mathcal{B}_t^{\mathbf{X}}, \quad t \in \mathbb{R}\}.$$

The vector field of the displacement of the motion of a material point p at time t is defined by $\mathbf{u}_{\mathbf{X}}(p, t) = \mathbf{X}(p, t) - p$ whereas the motion gradient is given by

$$F_{\mathbf{X}}(p, t) := \operatorname{grad} \mathbf{X}(p, t).$$

Immediately, we have $F_{\mathbf{X}}(p, t) = I + \operatorname{grad} \mathbf{u}_{\mathbf{X}}(p, t)$.

For a fixed time value t , the inverse mapping of the motion at time t , that is, $\mathbf{X}(\cdot, t)$, is denoted by $\mathbf{P}_X(\cdot, t) : \mathcal{B}_t \rightarrow \mathcal{B}$. Hence, this mapping,

$$\mathbf{P}_X : \mathcal{T}_{\mathbf{X}} \rightarrow \mathcal{B},$$

called the reference map of motion $\mathbf{X}$, computes the material point located at the spatial position x at time t .

Finally, let us introduce the time derivatives of a motion: the material velocity field of the material point p at time t is defined as the first time derivative of the motion $\mathbf{X}$,

$$\frac{\partial \mathbf{X}}{\partial t}(p, t),$$

and analogously, the material acceleration field of the material point p at time t is defined as the second time derivative of the motion $\mathbf{X}$,

$$\frac{\partial^2 \mathbf{X}}{\partial t^2}(p, t).$$

Scalar, vector or tensor fields defined in the trajectory of the motion $\mathcal{T}_{\mathbf{X}}$ are called spatial (Eulerian) fields because it depends on the spatial locations where they are evaluated. Those fields defined in $\mathcal{B} \times \mathbb{R}$ are called material (Lagrangian) fields because they are defined on the points of the reference configuration. By using the motion $\mathbf{X}$ and the reference map $\mathbf{P}_{\mathbf{X}}$, any spatial field can be transformed into a material field (composing it with the motion) and reciprocally, any material field can be rewritten as a spatial field (composing it with the reference map). In particular, it will be very relevant for the description of the proposed models to compute the spatial description of the material velocity, which is defined by

$$\mathbf{v}_{\mathbf{X}}(x, t) = \frac{\partial \mathbf{X}}{\partial t}(\mathbf{P}_{\mathbf{X}}(x, t), t).$$

Finally, we have to take into account that a spatial field can be derived in time with respect to the reference configuration. This kind of time derivative computations are essential for the statement of the equation of motion of the models described in this chapter. More precisely, the material time derivative of a spatial (scalar, vector or tensor valued) field will be denoted by $(\dot{\cdot})$ and also by D/Dt . For example, for a smooth spatial vector field $\mathbf{w} : \mathcal{T}_{\mathbf{X}} \rightarrow \mathcal{V}$, its material time derivative is defined by

$$\dot{\mathbf{w}}(x, t) = \left. \frac{\partial}{\partial t} (\mathbf{w}(\mathbf{X}(p, t), t)) \right|_{p=\mathbf{P}_{\mathbf{X}}(x, t)} = \frac{\partial \mathbf{w}}{\partial t}(x, t) + (\text{grad } \mathbf{w}(x, t)) \mathbf{v}_{\mathbf{X}}(x, t).$$

4.3 Motion equation for porous materials

A porous material with solid rigid frame has two main physical characteristics which determine the mass of the porous material: the fluid mass density distribution, which is filling the void micro-pore structure of the material and the porosity of the material, which is defined as the ratio of the fluid volume with respect to the total volume (including the fluid and the solid part) of the material. Consequently, we introduce as known data, the mass density of the fluid which fills the pores of the porous material placed on region $\mathcal{B}$, defined by the fluid mass density field $\rho_0 : \mathcal{B} \rightarrow \mathbb{R}^+$ in the reference configuration. In addition, we assume the scalar value of the porosity ϕ as a known quantity, which controls the portion of fluid mass involved inside the region $\mathcal{B}$. Throughout the derivation of the model, it will be assumed that the porosity ϕ is a constant value.

The mass conservation law for the amount of fluid contained in $\mathcal{B}$ implies that the transport of the spatial porous mass density $\rho_{\mathbf{X}}(x, t)$ due to the motion $\mathbf{X}$ satisfies

$$\rho_{\mathbf{X}}(x, t) \det F_{\mathbf{X}}(p, t) = \phi \rho_0(p), \quad x = \mathbf{X}(p, t), \quad (4.3.1)$$

which is equivalent in assuming the conservation of fluid mass between reference configuration $\mathcal{B}$ and deformed configuration $\mathcal{B}_t^{\mathbf{X}}$ at time t along the trajectory of the motion $\mathbf{X}$.

If $\mathcal{N}$ denotes the set of unit vectors of the linear space $\mathcal{V}$, then a system of forces associated with the motion can be introduced as follows.

Definition 4.3.1. *A system of forces applied on a body $\mathcal{B}$ during a motion $\mathbf{X}$ is given by a vector-valued mapping of boundary loads*

$$\mathbf{f}_{\mathbf{X}} : \mathcal{N} \times \mathcal{T}_{\mathbf{X}} \rightarrow \mathcal{V}$$

and a vector-valued mapping of volume forces

$$\mathbf{b}_{\mathbf{X}} : \mathcal{T}_{\mathbf{X}} \rightarrow \mathcal{V},$$

which satisfy the following assumptions:

- (i) *For each fixed vector $\mathbf{n} \in \mathcal{N}$ and time t , the application $x \mapsto \mathbf{f}_{\mathbf{X}}(\mathbf{n}, x, t)$ is a smooth function for all $x \in \mathcal{B}_t^{\mathbf{X}}$.*
- (ii) *For each fixed time t , the application $x \mapsto \mathbf{b}_{\mathbf{X}}(x, t)$ is continuous for all $x \in \mathcal{B}_t^{\mathbf{X}}$.*

The spatial vector field $\mathbf{f}_{\mathbf{X}}$ models the density of surface forces (force per unit area) across an oriented surface $\mathcal{S} \subset \mathcal{B}_t^{\mathbf{X}}$ with positive unit normal vector $\mathbf{n}$ at the spatial point x of the porous material and at time t , between the positive side of $\mathcal{S}$ and the negative side (the so-called Cauchy's hypothesis [46, Section 19.2]). The vector field $\mathbf{b}_{\mathbf{X}}$ accounts for a body force density (force per unit of volume), which models the action of the environment on the body at spatial point x and at time t in the deformed configuration $\mathcal{B}_t^{\mathbf{X}}$.

4.3.1 Motion equation in Eulerian coordinates

The classical balance of linear and angular momentum do not hold due to the dissipative character of the porous material with rigid solid frame. To account for that phenomena, Darcy and Forchheimer terms must be added to the classical balances. In the case of the Darcy term, the tensor α models a linear dissipative behaviour of the model with respect to the spatial velocity (related to the flow resistivity tensor of the standard linear model). Additionally, the Forchheimer correction includes a non-linear dissipative term which depends on the tensor β and it is proportional to the cubic power of the spatial description of the material velocity (see [70]). For simplicity in the derivation of the proposed porous model, both tensors α and β will be assumed constant and independent of the spatial points.

Hence, after introducing these modification we assume the following balances of linear

and angular momentum written at any time t and any arbitrary part $\mathcal{P}_t^{\mathbf{X}}$ of the trajectory:

$$\begin{aligned} & \frac{d}{dt} \int_{\mathcal{P}_t^{\mathbf{X}}} \rho_{\mathbf{X}}(x, t) \mathbf{v}_{\mathbf{X}}(x, t) dV_x + \underbrace{\int_{\mathcal{P}_t^{\mathbf{X}}} \rho_{\mathbf{X}}(x, t) \alpha \mathbf{v}_{\mathbf{X}}(x, t) dV_x}_{\text{Darcy term}} \\ & + \underbrace{\int_{\mathcal{P}_t^{\mathbf{X}}} \rho_{\mathbf{X}}(x, t) \|\mathbf{v}_{\mathbf{X}}(x, t)\| \beta \mathbf{v}_{\mathbf{X}}(x, t) dV_x}_{\text{Forchheimer term}} \\ & = \int_{\partial \mathcal{P}_t^{\mathbf{X}}} \mathbf{f}_{\mathbf{X}}(\mathbf{n}, x, t) dA_x + \int_{\mathcal{P}_t^{\mathbf{X}}} \mathbf{b}_{\mathbf{X}}(x, t) dV_x, \end{aligned} \quad (4.3.2)$$

$$\begin{aligned} & \frac{d}{dt} \int_{\mathcal{P}_t^{\mathbf{X}}} \rho_{\mathbf{X}}(x, t) \mathbf{r}(x) \times \mathbf{v}_{\mathbf{X}} dV_x + \underbrace{\int_{\mathcal{P}_t^{\mathbf{X}}} \rho_{\mathbf{X}}(x, t) \mathbf{r}(x) \times \alpha \mathbf{v}_{\mathbf{X}}(x, t) dV_x}_{\text{Darcy term}} \\ & + \underbrace{\int_{\mathcal{P}_t^{\mathbf{X}}} \rho_{\mathbf{X}}(x, t) \|\mathbf{v}_{\mathbf{X}}(x, t)\| \mathbf{r}(x) \times \beta \mathbf{v}_{\mathbf{X}}(x, t) dV_x}_{\text{Forchheimer term}} \\ & = \int_{\partial \mathcal{P}_t^{\mathbf{X}}} \mathbf{r}(x) \times \mathbf{f}_{\mathbf{X}}(\mathbf{n}, x, t) dA_x + \int_{\mathcal{P}_t^{\mathbf{X}}} \mathbf{r}(x) \times \mathbf{b}_{\mathbf{X}}(x, t) dV_x, \end{aligned} \quad (4.3.3)$$

where the arbitrary part $\mathcal{P}_t^{\mathbf{X}}$ of the trajectory is computed as the image of an arbitrary part $\mathcal{P}$ of the reference configuration using the motion $\mathbf{X}$, that is, $\mathcal{P}_t^{\mathbf{X}} = \mathbf{X}(\mathcal{P}, t)$. Notice that the vector position $\mathbf{r}(x) = x - o$ of a point x is computed with respect to the origin o fixed in the spatial affine space $\mathcal{E}$.

Obviously, the terms on the left-hand side of both equations are volume contributions to the balance of forces and momentum. Following [46, Section 19.4] all those terms can be included in a unique generalised body force $\mathbf{b}_{\mathbf{X}}$ and consequently proceed in a standard way to prove the classical Cauchy's Theorem. For this purpose, we introduce $E(\mathcal{V})$, which is the linear space of endomorphisms of $\mathcal{V}$ (which is isomorphic to the space of second order tensors) and also $E_{\text{sym}}(\mathcal{V})$, which denotes the subspace of the symmetric maps. Following the classical arguments for the non-dissipative balance of forces and moments (see [46, Section 19.5]), the Cauchy's Theorem guarantees there exists a smooth symmetric second order tensor field

$$T_{\mathbf{X}} : \mathcal{T}_{\mathbf{X}} \rightarrow E_{\text{sym}}(\mathcal{V}),$$

called the Cauchy stress tensor, such that

$$\mathbf{f}_{\mathbf{X}}(\mathbf{n}, x, t) = T_{\mathbf{X}}(x, t) \mathbf{n},$$

for all $(x, t) \in \mathcal{T}_{\mathbf{X}}$ and all $\mathbf{n} \in \mathcal{N}$.

To obtain the motion governing equation associated with motion which satisfies the balance of forces and moments described above, we must prove a result which is an immediate consequence of the Reynolds' transport Theorem (see [46, Chapter 16]).

Proposition 4.3.1. *Let Φ be a smooth spatial vector field, $\Phi : \mathcal{T}_{\mathbf{X}} \rightarrow \mathcal{V}$. For any time t and any subregion $\mathcal{P}_{\mathbf{X}}^t \subset \mathcal{B}_{\mathbf{X}}^t$, it holds*

$$\frac{d}{dt} \int_{\mathcal{P}_{\mathbf{X}}^t} \rho_{\mathbf{X}}(x, t) \Phi(x, t) dV_x = \int_{\mathcal{P}_{\mathbf{X}}^t} \rho_{\mathbf{X}}(x, t) \dot{\Phi}(x, t) dV_x.$$

Now, using Prop. 4.3.1 with $\Phi = \mathbf{v}_{\mathbf{X}}$ to rewrite the first term in (4.3.2), after the substitution of the boundary loads by using the Cauchy stress tensor, and applying a localisation argument (see [46, Section 15.3]) for the integrals involved in the balance of forces (4.3.2), it holds

$$\rho_{\mathbf{X}} (\dot{\mathbf{v}}_{\mathbf{X}} + \alpha \mathbf{v}_{\mathbf{X}} + \|\mathbf{v}_{\mathbf{X}}\| \beta \mathbf{v}_{\mathbf{X}}) = \operatorname{div} T_{\mathbf{X}} + \mathbf{b}_{\mathbf{X}}. \quad (4.3.4)$$

The motion equation (4.3.4) is valid for any Cauchy stress tensor T . However, it is widely used in the mathematical modelling of porous materials with rigid solid frame to assume that the governing equation of porous materials is given by a fluid-equivalent model (see [6]). To introduce such fluid-like models, some constitutive equations for the porous material with rigid solid frame must be assumed, considering that the motion is isothermal process and neither viscous nor shear forces are involved in the computation of the Cauchy stress tensor, this is:

(H1) There exists a smooth spatial scalar field $\pi_{\mathbf{X}} : \mathcal{T}_{\mathbf{X}} \rightarrow \mathbb{R}$ called thermodynamic pressure such that

$$T_{\mathbf{X}}(x, t) = -\pi_{\mathbf{X}}(x, t) I$$

for all $(x, t) \in \mathcal{T}_{\mathbf{X}}$, where I is the identity tensor.

(H2) There exists a smooth response function $\hat{\pi} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\pi_{\mathbf{X}}(x, t) = \hat{\pi}(\rho_{\mathbf{X}}(x, t), \theta_{\mathbf{X}}(x, t)), \quad x = \mathbf{X}(p, t),$$

where $\theta_{\mathbf{X}}$ denotes the temperature.

(H3) The response function $\hat{\pi}$ of the pressure field can be written in terms of the entropy s and the temperature θ and it is satisfied

$$\frac{\partial \hat{\pi}}{\partial \rho}(\rho, s) = \gamma \frac{\partial \hat{\pi}}{\partial \rho}(\rho, \theta),$$

being γ the ratio of specific heats at constant pressure and volume (see [19, Lemma 10.4.14]).

Consequently, the equation of motion (4.3.4) can be rewritten in terms of the pressure field as follows:

$$\rho_{\mathbf{X}} (\dot{\mathbf{v}}_{\mathbf{X}} + \alpha \mathbf{v}_{\mathbf{X}} + \|\mathbf{v}_{\mathbf{X}}\| \beta \mathbf{v}_{\mathbf{X}}) = \operatorname{grad} \pi_{\mathbf{X}} + \mathbf{b}_{\mathbf{X}}.$$

4.3.2 Motion equation in Lagrangian coordinates

Once the equation of motion has been described on the trajectory (deformed configuration) and by means of spatial (Eulerian) coordinates, this governing equation can be also written using material (Lagrangian) coordinates. Consequently, the new version of the motion equation will be stated in a fixed domain, that is, in the reference configuration $\mathcal{B}$.

With this aim, a classical result for change of variables involving surface integrals will be used (see [46, Section 15.2]).

Proposition 4.3.2. *Let Ψ be a continuous tensor field, $\Psi : \mathcal{T}_{\mathbf{X}} \rightarrow E(\mathcal{V})$. For a given motion $\mathbf{X}$, a fixed time t , and any arbitrary part $\mathcal{P} \subset \mathcal{B}$, it holds*

$$\int_{\partial \mathcal{P}_t^{\mathbf{X}}} S(x, t) \mathbf{n}(x, t) dA_x = \int_{\partial \mathcal{P}} \det F_{\mathbf{X}}(p, t) S(\mathbf{X}(p, t), t) F_{\mathbf{X}}^{-t}(p, t) \mathbf{m}(p) dA_p,$$

where recall $\mathcal{P}_t^{\mathbf{X}} = \mathbf{X}(\mathcal{P}, t)$, $\mathbf{n}(x, t)$ denotes an outward unit normal vector to $\partial \mathcal{P}_t^{\mathbf{X}}$ at point x and time t , and $\mathbf{m}(p)$ denotes an outward unit normal vector to $\partial \mathcal{P}$ at point p .

Now, if the Cauchy stress tensor is introduced in the balance of forces law (4.3.2), the change of variable by means of the motion, i.e., $x = \mathbf{X}(p, t)$, and the use of Prop. 4.3.2 leads to

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{P}} \phi \rho_0(p) \frac{\partial \mathbf{X}}{\partial t}(p, t) dV_p + \int_{\mathcal{P}} \phi \rho_0(p) \alpha \frac{\partial \mathbf{X}}{\partial t}(p, t) dV_p \\ + \int_{\mathcal{P}} \phi \rho_0(p) \left\| \frac{\partial \mathbf{X}}{\partial t}(p, t) \right\| \beta \frac{\partial \mathbf{X}}{\partial t}(p, t) dV_p \\ = \int_{\partial \mathcal{P}} S_{\mathbf{X}}(p, t) \mathbf{m}(p) dA_p + \int_{\mathcal{P}} \mathbf{b}_{\mathbf{X}}^*(p, t) dV_p, \end{aligned} \quad (4.3.5)$$

where the body force contribution at the reference configuration is given by

$$\mathbf{b}_{\mathbf{X}}^*(p, t) = \det F_{\mathbf{X}}(p, t) \mathbf{b}_{\mathbf{X}}(\mathbf{X}(p, t), t),$$

and $S_{\mathbf{X}}(p, t)$ is the first Piola-Kirchhoff stress tensor defined in the reference configuration by

$$S_{\mathbf{X}}(p, t) := \det F_{\mathbf{X}}(p, t) T_{\mathbf{X}}(x, t) F_{\mathbf{X}}^{-t}(p, t), \quad x = \mathbf{X}(p, t).$$

Thanks to the use of a Green's formula and a localization argument, the equation of motion can be written in Lagrangian coordinates as follows:

$$\phi \rho_0 \left(\frac{\partial^2 \mathbf{X}}{\partial t^2} + \alpha \frac{\partial \mathbf{X}}{\partial t} + \left\| \frac{\partial \mathbf{X}}{\partial t} \right\| \beta \frac{\partial \mathbf{X}}{\partial t} \right) = \operatorname{div}_p S_{\mathbf{X}} + \mathbf{b}_{\mathbf{X}}^*. \quad (4.3.6)$$

Let us remark that now this partial differential equations (4.3.6) is stated in a fixed (time-independent) domain $\mathcal{B} \times (0, \infty)$.

4.4 Modelling small perturbations from an underlying flow

Once the proposed motion of equation with the Darcy Forchheimer terms has been written in Eulerian and Lagrangian coordinates, we are in a position to derive a linear model, which will govern the small perturbations from an underlying motion (possibly with a different scale). The underlying given motion is denoted by $\mathbf{Y}$ and by $\mathbf{X}$ accounts for the motion associated with a small isothermal perturbation of it. Hence, the spatial temperature field $\theta_{\mathbf{X}}$ holds

$$\theta_{\mathbf{X}}(\mathbf{X}(p, t), t) = \theta_{\mathbf{Y}}(\mathbf{Y}(p, t), t) \quad \forall (p, t) \in \mathcal{B} \times \mathbb{R}. \quad (4.4.1)$$

This equality establishes that the material points p conserves the temperature along time trajectory produced by the motion $\mathbf{X}$. However, the isothermal character of the small perturbation $\mathbf{X}$ has not to be inherited by the motion $\mathbf{Y}$, which could be itself non-isothermal.

For example, motion $\mathbf{Y}$ could be considered as a fluid flow through the porous material and motion $\mathbf{X}$ could be read as a small perturbation of the motion $\mathbf{Y}$ produced by the propagation of an isothermal (compression) acoustic wave. Hence, this section is devoted to write a linear partial differential system which governs the small perturbations of the motion $\mathbf{X}$.

With this aim, firstly we need to understand the small perturbation motion $\mathbf{X}$ as the composition of two different motions: the motion $\mathbf{Y}$ and the motion $\mathbf{Z} : \mathcal{T}_{\mathbf{Y}} \rightarrow \mathcal{E}$ defined by

$$\mathbf{Z}(y, t) = \mathbf{X}(\mathbf{P}_{\mathbf{Y}}(y, t), t) \quad \forall (y, t) \in \mathcal{T}_{\mathbf{Y}}. \quad (4.4.2)$$

Hence, motion $\mathbf{Z}$ is the motion $\mathbf{X}$ rewritten in terms of the Eulerian coordinates associated with the motion $\mathbf{Y}$. From (4.4.2), if we rewrite it in terms of material points using $y = \mathbf{Y}(p, t)$ and $p = \mathbf{P}_{\mathbf{Y}}(y, t)$, we have

$$\mathbf{X}(p, t) = \mathbf{Z}(\mathbf{Y}(p, t), t) \quad \forall (p, t) \in \mathcal{B} \times \mathbb{R}.$$

and hence, computing the gradients of both side of the equation above, it leads to

$$F_{\mathbf{X}}(p, t) = F_{\mathbf{Z}}(\mathbf{Y}(p, t), t) F_{\mathbf{Y}}(p, t), \quad (4.4.3)$$

where $F_{\mathbf{Z}}(y, t) = \text{grad}_y(\mathbf{Z}(y, t))$. With this composition of motions in mind, the integral formulation of the balance of forces in Lagrangian coordinates (4.3.5) can be rewritten using

the change of variable $y = \mathbf{Y}(p, t)$. Then, we obtain

$$\begin{aligned}
 & \frac{d}{dt} \int_{\mathcal{P}_t^{\mathbf{Y}}} \frac{\phi \rho_0(\mathbf{P}_{\mathbf{Y}}(y, t))}{\det F_{\mathbf{Y}}(\mathbf{P}_{\mathbf{Y}}(y, t), t)} \frac{\partial \mathbf{X}}{\partial t}(\mathbf{P}_{\mathbf{Y}}(y, t), t) dV_y \\
 & \quad + \int_{\mathcal{P}_t^{\mathbf{Y}}} \frac{\phi \rho_0(\mathbf{P}_{\mathbf{Y}}(y, t))}{\det F_{\mathbf{Y}}(\mathbf{P}_{\mathbf{Y}}(y, t), t)} \alpha \frac{\partial \mathbf{X}}{\partial t}(\mathbf{P}_{\mathbf{Y}}(y, t), t) dV_y \\
 & \quad + \int_{\mathcal{P}_t^{\mathbf{Y}}} \frac{\phi \rho_0(\mathbf{P}_{\mathbf{Y}}(y, t))}{\det F_{\mathbf{Y}}(\mathbf{P}_{\mathbf{Y}}(y, t), t)} \left\| \frac{\partial \mathbf{X}}{\partial t}(\mathbf{P}_{\mathbf{Y}}(y, t), t) \right\| \beta \frac{\partial \mathbf{X}}{\partial t}(\mathbf{P}_{\mathbf{Y}}(y, t), t) dV_y \\
 & = - \int_{\partial \mathcal{P}_t^{\mathbf{Y}}} \frac{\det F_{\mathbf{X}}(\mathbf{P}_{\mathbf{Y}}(y, t), t)}{\det F_{\mathbf{Y}}(\mathbf{P}_{\mathbf{Y}}(y, t), t)} \hat{\pi}(\rho_{\mathbf{X}}(\mathbf{Z}(y, t), t), \theta_{\mathbf{X}}(\mathbf{Z}(y, t), t)) F_{\mathbf{X}}^{-t}(\mathbf{P}_{\mathbf{Y}}(y, t), t) \\
 & \quad F_{\mathbf{Y}}^t(\mathbf{P}_{\mathbf{Y}}(y, t), t) \boldsymbol{\nu}(y, t) dA_y + \int_{\mathcal{P}_t^{\mathbf{Y}}} \frac{\det F_{\mathbf{X}}(\mathbf{P}_{\mathbf{Y}}(y, t), t)}{\det F_{\mathbf{Y}}(\mathbf{P}_{\mathbf{Y}}(y, t), t)} \mathbf{b}_{\mathbf{X}}(\mathbf{Z}(y, t)) dV_y, \quad (4.4.4)
 \end{aligned}$$

where $\boldsymbol{\nu}(y, t)$ is the outward unit normal vector to $\partial \mathcal{P}_t^{\mathbf{Y}}$ at point y and time t .

Moreover, from the mass conservation equation (4.3.1), applied to the motion $\mathbf{Y}$ (notice that both motions $\mathbf{X}$ and $\mathbf{Y}$ satisfy the assumption about the conservation of mass) and using (4.4.3) to express $F_{\mathbf{Y}}$ in terms of $F_{\mathbf{X}}$ and $F_{\mathbf{Z}}$, we have

$$\rho_{\mathbf{Y}}(y, t) = \frac{\phi \rho_0(\mathbf{P}_{\mathbf{Y}}(y, t))}{\det F_{\mathbf{Y}}(\mathbf{P}_{\mathbf{Y}}(y, t), t)}, \quad (4.4.5)$$

$$\det F_{\mathbf{Z}}(y, t) = \frac{\det F_{\mathbf{X}}(\mathbf{P}_{\mathbf{Y}}(y, t), t)}{\det F_{\mathbf{Y}}(\mathbf{P}_{\mathbf{Y}}(y, t), t)}. \quad (4.4.6)$$

Taking into account the isothermal character of the motion $\mathbf{X}$ (4.4.1) and the relation between gradients (4.4.3), and since

$$\mathbf{v}_{\mathbf{X}}(\mathbf{Z}(y, t), t) = \frac{\partial \mathbf{X}}{\partial t}(\mathbf{P}_{\mathbf{Y}}(y, t), t),$$

then the balance of forces for motion $\mathbf{X}$ at the deformed configuration associated with the motion $\mathbf{Y}$ (4.4.4) becomes

$$\begin{aligned}
 & \frac{d}{dt} \int_{\mathcal{P}_t^{\mathbf{Y}}} \rho_{\mathbf{Y}}(y, t) \mathbf{v}_{\mathbf{X}}(\mathbf{Z}(y, t), t) dV_y + \int_{\mathcal{P}_t^{\mathbf{Y}}} \rho_{\mathbf{Y}}(y, t) \alpha \mathbf{v}_{\mathbf{X}}(\mathbf{Z}(y, t), t) dV_y \\
 & \quad + \int_{\mathcal{P}_t^{\mathbf{Y}}} \rho_{\mathbf{Y}}(y, t) \|\mathbf{v}_{\mathbf{X}}(\mathbf{Z}(y, t), t)\| \beta \mathbf{v}_{\mathbf{X}}(\mathbf{Z}(y, t), t) dV_y \\
 & = - \int_{\partial \mathcal{P}_t^{\mathbf{Y}}} \det F_{\mathbf{Z}}(y, t) \hat{\pi}(\rho_{\mathbf{X}}(\mathbf{Z}(y, t), t), \theta_{\mathbf{Y}}(y, t)) F_{\mathbf{Z}}^{-t}(y, t) \boldsymbol{\nu}(y) dA_y \\
 & \quad + \int_{\mathcal{P}_t^{\mathbf{Y}}} \det F_{\mathbf{Z}}(y, t) \mathbf{b}_{\mathbf{X}}(\mathbf{Z}(y, t), t) dV_y. \quad (4.4.7)
 \end{aligned}$$

Besides the force conservation statement written above, the balance of forces satisfied by the motion $\mathbf{Y}$ itself yields to (see (4.3.2) and the Cauchy's theorem but now applied to motion $\mathbf{Y}$) leads to

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{P}_t^Y} \rho_Y(y, t) \mathbf{v}_Y(y, t) dV_y + \int_{\mathcal{P}_t^Y} \rho_Y(y, t) (\alpha + \|\mathbf{v}_Y(y, t)\| \beta) \mathbf{v}_Y(y, t) dV_y \\ = \int_{\partial \mathcal{P}_t^Y} T_Y(y, t) \boldsymbol{\nu}(y, t) dA_y + \int_{\mathcal{P}_t^Y} \mathbf{b}_Y(y, t) dV_y. \end{aligned} \quad (4.4.8)$$

Since motion $\mathbf{X}$ is considered as a small perturbation from the motion described by $\mathbf{Y}$, the difference of the spatial position given by both motions should be small. Hence, it is natural to introduce the vector field $\mathbf{w}$ of the perturbed displacement as a function of Eulerian variables of motion $\mathbf{Y}$ given by

$$\mathbf{w}(y, t) = \mathbf{Z}(y, t) - y, \quad (4.4.9)$$

and its material description $\mathbf{w}_m$ in material (Lagrangian) coordinates which satisfies

$$\mathbf{w}_m(p, t) = \mathbf{w}(\mathbf{Y}(p, t), t) = \mathbf{Z}(\mathbf{Y}(p, t), t) - \mathbf{Y}(p, t) = \mathbf{X}(p, t) - \mathbf{Y}(p, t). \quad (4.4.10)$$

In addition, the following equalities hold, which relate $\mathbf{w}$ and $\mathbf{w}_m$ with the motions $\mathbf{X}$ and $\mathbf{Y}$.

Lemma 4.4.1. *Let the motions $\mathbf{X}$ and $\mathbf{Y}$ satisfy (4.4.2) and the perturbed displacement be defined by (4.4.9). It holds*

$$\text{grad}_p \mathbf{w}_m(p, t) = F_{\mathbf{X}}(p, t) - F_{\mathbf{Y}}(p, t), \quad \text{for all } (p, t) \in \mathcal{B} \times \mathbb{R}, \quad (4.4.11)$$

$$\text{grad}_y \mathbf{w}(y, t) = F_{\mathbf{Z}}(y, t) - I, \quad (4.4.12)$$

$$\dot{\mathbf{w}}(y, t) = \frac{\partial \mathbf{X}}{\partial t}(\mathbf{P}_Y(y, t), t) - \mathbf{v}_Y(y, t) = \mathbf{v}_X(\mathbf{Z}(y, t), t) - \mathbf{v}_Y(y, t), \quad (4.4.13)$$

for all $(y, t) \in \mathcal{T}_Y$, and where $\dot{\mathbf{w}}$ is the material time derivative of $\mathbf{w}$ with respect to motion $\mathbf{Y}$, i.e. $\dot{\mathbf{w}}(y, t) = \partial \mathbf{w} / \partial t + (\text{grad}_y \mathbf{w}) \mathbf{v}_Y$.

Proof. See Lemma 3.1 in [18]. □

Remark 4.4.1. *In what follows, the subscripts on the differential operators will be omitted and it will be assumed that the differential operators are acting with respect to the variables which are used as arguments of the fields involved in the differentiation.*

If Equation (4.4.8) is subtracted from (4.4.7), the time material derivative $\dot{\mathbf{w}}$ is rewritten using the third equality in Lemma 4.4.1, and the constitutive laws **(H1)** and **(H2)** are used

to replace the Cauchy stress tensor by the response function of the pressure field, we obtain

$$\begin{aligned}
 & \frac{d}{dt} \int_{\mathcal{P}_t^Y} \rho_Y(y, t) \dot{\mathbf{w}}(y, t) dV_y + \int_{\mathcal{P}_t^Y} \rho_Y(y, t) \alpha \dot{\mathbf{w}}(y, t) dV_y \\
 & + \int_{\mathcal{P}_t^Y} \rho_Y(y, t) \beta (\|\mathbf{v}_X(Z(y, t), t)\| \mathbf{v}_X(Z(y, t), t) - \|\mathbf{v}_Y(y, t)\| \mathbf{v}_Y(y, t)) dV_y \\
 & = - \int_{\partial \mathcal{P}_t^Y} [\det F_Z(y, t) \hat{\pi}(\rho_X(\mathbf{Z}(y, t), t), \theta_Y(y, t)) F_Z^{-t}(y, t) \\
 & \quad - \hat{\pi}(\rho_Y(y, t), \theta_Y(y, t)) I] \nu(y) dA_y \\
 & \quad + \int_{\mathcal{P}_t^Y} [\det F_Z(y, t) \mathbf{b}_X(\mathbf{Z}(y, t), t) - \mathbf{b}_Y(y, t)] dV_y. \quad (4.4.14)
 \end{aligned}$$

To obtain a linear approximation of the balance of forces stated in the deformed configuration $\mathcal{P}_t^Y$ and written in terms of $\mathbf{w}$, we need to get a linear approximation of the third term in the left-hand side and the first term of the right-hand side in (4.4.14). First, let us consider the boundary term in the right-hand side. Since the pressure field can be considered as a function of the motion gradient F_X , then it can be approximated in a neighbourhood of F_Y by using its Fréchet derivative.

For this linearization purpose, We will use repeatedly of the following differential result about the determinant of a second-order invertible tensor (see for instance [46]). In what follows, the subspace of $E(\mathcal{V})$ containing the non-singular tensors will be denoted by $I(\mathcal{V})$.

Lemma 4.4.2. *Let A be a non-singular second-order tensor in $I(\mathcal{V})$. If φ is given by $\varphi(A) = \det A$, then φ is differentiable and it holds*

$$\frac{\partial \varphi}{\partial A}(A)(U) = \det A \operatorname{tr}(U A^{-1})$$

for all $U \in E(\mathcal{V})$.

Since the response function of the pressure field $\hat{\pi}$ is assumed also smooth, for a fixed non-singular tensor $H \in I(\mathcal{V})$ and a given temperature θ , the mapping $\hat{g}_H : I(\mathcal{V}) \rightarrow \mathbb{R}$ defined by

$$\hat{g}_H(F) = \hat{\pi}\left(\frac{\phi \rho_0}{\det(FH)}, \theta\right) \quad (4.4.15)$$

is also Fréchet differentiable and its differential operator can be computed as follows.

Lemma 4.4.3. *For a fixed tensor $H \in I(\mathcal{V})$ and a given temperature θ , the differential operator of the mapping $\hat{g}_H$ defined by (4.4.15) is given by*

$$\frac{\partial \hat{g}_H}{\partial F}(F)(U) = - \frac{\phi \rho_0}{\det(FH)} \frac{\partial \hat{\pi}}{\partial \rho} \left(\frac{\phi \rho_0}{\det(FH)}, \theta \right) \operatorname{tr}(U F^{-1}).$$

Proof. A straightforward application of the chain rule and Lemma 4.4.2 yields to the final result. $\square$

Thanks to Lemma 4.4.3, we are in a position to compute the differential operator of the integrand of the first term in the right-hand side of (4.4.14). Before proceeding with the linearization of that term, we also need the following auxiliary result.

Proposition 4.4.1. *For a fixed tensor $H \in I(\mathcal{V})$ and a given temperature θ , the mapping $\hat{G}_H : I(\mathcal{V}) \rightarrow \mathbb{R}$, defined by*

$$\hat{G}_H(F) = \det F \hat{\pi} \left(\frac{\phi \rho_0}{\det(FH)}, \theta \right) F^{-t},$$

is Fréchet differentiable and its derivative is given by

$$\begin{aligned} \frac{\partial \hat{G}_H}{\partial F}(F)(U) &= -\frac{\phi \rho_0}{\det(FH)} \frac{\partial \hat{\pi}}{\partial \rho} \left(\frac{\phi \rho_0}{\det(FH)}, \theta \right) \text{tr}(UF^{-1}) F^{-t} \\ &\quad + \det F \text{tr}(UF^{-1}) \hat{\pi} \left(\frac{\phi \rho_0}{\det(FH)}, \theta \right) F^{-t} - \det F \hat{\pi} \left(\frac{\phi \rho_0}{\det(FH)}, \theta \right) U^t F^{-t}. \end{aligned} \quad (4.4.16)$$

Proof. The definition of $\hat{G}_H(F)$ implies

$$\hat{G}_H(F)F^t = \det F \hat{\pi} \left(\frac{\phi \rho_0}{\det(FH)}, \theta \right) I$$

and hence we have

$$\begin{aligned} \frac{\partial \hat{G}_H}{\partial F}(F)(U)F^t + \hat{G}_H(F)U^t &= \det F \text{tr}(UF^{-1}) \hat{\pi} \left(\frac{\phi \rho_0}{\det(FH)}, \theta \right) I \\ &\quad - \frac{\phi \rho_0}{\det(FH)} \frac{\partial \hat{\pi}}{\partial \rho} \left(\frac{\phi \rho_0}{\det(FH)}, \theta \right) \text{tr}(UF^{-1}) I, \end{aligned}$$

from which the result follows. $\square$

Now, we are in a position to approximate the integrand of the first term in the right-hand side of (4.4.14) with a linear expression in terms of the displacement field $\mathbf{w}$.

Proposition 4.4.2. *Let the motions $\mathbf{X}$ and $\mathbf{Y}$ satisfy (4.4.2) and the perturbed displacement $\mathbf{w}$ be defined by (4.4.9). If their respective mass conservation laws (4.3.1) and (4.4.5) are satisfied then we have*

$$\begin{aligned} &\det F_{\mathbf{Z}}(y, t) \hat{\pi}(\rho_{\mathbf{X}}(\mathbf{Z}(y, t), t), \theta_{\mathbf{Y}}(y, t)) F_{\mathbf{Z}}^{-t}(y, t) - \hat{\pi}(\rho_{\mathbf{Y}}(y, t), \theta_{\mathbf{Y}}(y, t)) I \\ &= -\rho_{\mathbf{Y}}(y, t) \frac{\partial \hat{\pi}}{\partial \rho}(\rho_{\mathbf{Y}}(y, t), \theta_{\mathbf{Y}}(y, t)) \text{div } \mathbf{w}(y, t) I + \text{div } \mathbf{w}(y, t) \hat{\pi}(\rho_{\mathbf{Y}}(y, t), \theta_{\mathbf{Y}}(y, t)) I \\ &\quad - \hat{\pi}(\rho_{\mathbf{Y}}(y, t), \theta_{\mathbf{Y}}(y, t)) \text{grad } \mathbf{w}^t(y, t) + o(\text{grad } \mathbf{w}(y, t)). \end{aligned} \quad (4.4.17)$$

Proof. Since $\mathbf{X}$ and $\mathbf{Y}$ are motions, their spatial gradients are non-singular and (4.4.2) yields to $F_{\mathbf{Z}}$ is invertible. Hence, the above Prop. 4.4.1 can be applied by choosing $F = F_{\mathbf{Z}}(y, t)$, $H = F_{\mathbf{Y}}(p, t)$ and $\theta = \theta_{\mathbf{Y}}(y, t)$. Then, using again (4.4.2), we obtain

$$\hat{G}_{F_{\mathbf{Z}}(p,t)}(F(y, t)) = \det F_{\mathbf{Z}}(y, t) \hat{\pi} \left(\frac{\phi \rho_0(p)}{\det(F_{\mathbf{X}}(p, t))}, \theta_{\mathbf{Y}}(y, t) \right) F_{\mathbf{Z}}^{-t}(y, t).$$

Since $\hat{G}_H$ is differentiable then it is straightforward to compute a linear approximation given by its first-order Taylor expansion, this is,

$$\hat{G}_H(F) = \hat{G}_H(I) + \frac{\partial \hat{G}_H}{\partial F}(I)(F - I) + o(F - I). \quad (4.4.18)$$

The first term in this expansion is given by

$$\hat{G}_{F_{\mathbf{Y}}(p,t)}(I) = \hat{\pi} \left(\frac{\phi \rho_0(p)}{\det(F_{\mathbf{Y}}(p, t))}, \theta_{\mathbf{Y}}(y, t) \right) I = \pi_{\mathbf{Y}}(y, t) I, \quad (4.4.19)$$

whereas, using the second equality in Lemma 4.4.1 and (4.4.5), the second term can be rewritten as follows:

$$\begin{aligned} & \frac{\partial \hat{G}_{F_{\mathbf{Y}}(p,t)}}{\partial F}(I) (F_{\mathbf{Z}}(y, t) - I) = \frac{\partial \hat{G}_{F_{\mathbf{Y}}(p,t)}}{\partial F}(I) (\text{grad } \mathbf{w}(y, t)) \\ &= - \frac{\phi \rho_0(p)}{\det F_{\mathbf{Y}}(p, t)} \frac{\partial \hat{\pi}}{\partial \rho} \left(\frac{\phi \rho_0(p)}{\det F_{\mathbf{Y}}(p, t)}, \theta_{\mathbf{Y}}(y, t) \right) \text{div } \mathbf{w}(y, t) I \\ & \quad + \text{div } \mathbf{w}(y, t) \hat{\pi} \left(\frac{\phi \rho_0(p)}{\det F_{\mathbf{Y}}(p, t)}, \theta_{\mathbf{Y}}(y, t) \right) I - \hat{\pi} \left(\frac{\phi \rho_0(p)}{\det F_{\mathbf{Y}}(p, t)}, \theta_{\mathbf{Y}}(y, t) \right) \text{grad } \mathbf{w}(y, t)^t \\ &= - \rho_{\mathbf{Y}}(y, t) \frac{\partial \hat{\pi}}{\partial \rho} (\rho_{\mathbf{Y}}(y, t), \theta_{\mathbf{Y}}(y, t)) \text{div } \mathbf{w}(y, t) I \\ & \quad + \text{div } \mathbf{w}(y, t) \hat{\pi} (\rho_{\mathbf{Y}}(y, t), \theta_{\mathbf{Y}}(y, t)) I - \hat{\pi} (\rho_{\mathbf{Y}}(y, t), \theta_{\mathbf{Y}}(y, t)) \text{grad } \mathbf{w}^t(y, t). \end{aligned} \quad (4.4.20)$$

Finally, the result is obtained by replacing (4.4.19) and (4.4.20) in (4.4.18) and taking into account that $F - I = F_{\mathbf{Z}}(y, t) - I = \text{grad } \mathbf{w}(y, t)$ (see the second equality in Lemma 4.4.1). $\square$

Proposition 4.4.3. *Let the motions $\mathbf{X}$ and $\mathbf{Y}$ satisfy (4.4.2) and the perturbed displacement $\mathbf{w}$ be defined by (4.4.9). If their respective mass conservation laws (4.3.1) and (4.4.5) are satisfied and the balance of forces (4.4.14) holds then we have*

$$\begin{aligned} & \rho_{\mathbf{Y}}(y, t) \ddot{\mathbf{w}}(y, t) + \rho_{\mathbf{Y}}(y, t) \alpha \dot{\mathbf{w}}(y, t) + \rho_{\mathbf{Y}}(y, t) \beta \|\mathbf{v}_{\mathbf{Y}}(y, t)\| \dot{\mathbf{w}}(y, t) \\ & \quad + \rho_{\mathbf{Y}}(y, t) \beta \frac{\mathbf{v}_{\mathbf{Y}}(y, t) \otimes \mathbf{v}_{\mathbf{Y}}(y, t)}{\|\mathbf{v}_{\mathbf{Y}}(y, t)\|} \dot{\mathbf{w}}(y, t) \\ &= \text{div} \left[\frac{\rho_{\mathbf{Y}}(y, t) c_{\mathbf{Y}}^2(y, t)}{\gamma} \text{div } \mathbf{w}(y, t) I + \text{div } \mathbf{w}(y, t) \pi_{\mathbf{Y}}(y, t) I - \pi_{\mathbf{Y}}(y, t) \text{grad } \mathbf{w}^t(y, t) \right] \\ & \quad + \det F_{\mathbf{Z}}(y, t) \mathbf{b}_{\mathbf{X}}(\mathbf{Z}(y, t), t) - \mathbf{b}_{\mathbf{Y}}(y, t) + o(\text{grad } \mathbf{w}(y, t)), \end{aligned} \quad (4.4.21)$$

where the sound speed $c_{\mathbf{Y}}(y, t)$ is locally defined by

$$c_{\mathbf{Y}}^2(y, t) = \gamma \frac{\partial \hat{\pi}}{\partial \rho}(\rho_{\mathbf{Y}}(y, t), \theta_{\mathbf{Y}}(y, t)), \quad (4.4.22)$$

being γ the ratio of specific heats at constant volume and pressure (see Hypothesis **(H3)**).

Proof. First, the linear approximation of the Forchheimer term in (4.4.14) must be provided. For that purpose, let us introduce the vector-valued mapping $\boldsymbol{\psi} : \mathcal{V} \rightarrow \mathcal{V}$ such that $\boldsymbol{\psi}(\mathbf{v}) = \|\mathbf{v}\|\mathbf{v}$. Proceeding in an analogous way as it has been done above, the linear approximation can be obtained by its first-order Taylor expansion as follows:

$$\boldsymbol{\psi}(\mathbf{v}) = \mathbf{f}(\mathbf{a}) + \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{v}}(\mathbf{a})(\mathbf{v} - \mathbf{a}) + o(\mathbf{v} - \mathbf{a}) \quad (4.4.23)$$

for all $\mathbf{v}, \mathbf{a} \in \mathcal{V}$. A simple computation of the partial derivatives of each component of $\boldsymbol{\psi}(\mathbf{v})$ shows that

$$\frac{\partial \psi_i}{\partial v_j}(\mathbf{v}) = \|\mathbf{v}\|\delta_{ij} + \frac{v_i}{\|\mathbf{v}\|}v_j, \quad 1 \leq i, j \leq 3,$$

where δ_{ij} is the Kronecker's delta. Hence, it holds $\partial \boldsymbol{\psi} / \partial \mathbf{v} = \|\mathbf{v}\|I + (\mathbf{v} \otimes \mathbf{v}) / \|\mathbf{v}\|$. Applying the linear approximation (4.4.23) to the Forchheimer term in (4.4.14) with $\mathbf{v} = \dot{\mathbf{w}}(y, t)$ and $\mathbf{a} = \mathbf{v}_{\mathbf{Y}}(y, t)$, and taking into account the last equality in Lemma 4.4.1, we obtain

$$\begin{aligned} & \|\mathbf{v}_{\mathbf{X}}(Z(y, t), t)\| \mathbf{v}_{\mathbf{X}}(Z(y, t), t) - \|\mathbf{v}_{\mathbf{Y}}(y, t)\| \mathbf{v}_{\mathbf{Y}}(y, t) \\ &= \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{v}}(\mathbf{v}_{\mathbf{Y}}(y, t)) \dot{\mathbf{w}}(y, t) = \|\mathbf{v}_{\mathbf{Y}}(y, t)\| \dot{\mathbf{w}}(y, t) + \frac{\mathbf{v}_{\mathbf{Y}}(y, t) \otimes \mathbf{v}_{\mathbf{Y}}(y, t)}{\|\mathbf{v}_{\mathbf{Y}}(y, t)\|} \dot{\mathbf{w}}(y, t). \end{aligned}$$

Finally, using the equation above and once it has been replaced (4.4.17) in (4.4.14), the use of Prop. 4.3.1 leads to the final result taking into account a standard Gauss' and localization arguments for any arbitrary part of the trajectory of motion $\mathbf{Y}$. $\square$

Following the arguments used in [18], the vector calculus formulas

$$\text{grad}(\varphi \xi) = \varphi \text{grad} \xi + \xi \text{grad} \varphi, \quad (4.4.24)$$

$$\text{div}(\varphi A) = A \text{grad} \varphi + \varphi \text{div} A, \quad (4.4.25)$$

$$\text{div}(\mathbf{a} \otimes \mathbf{c}) = (\text{grad} \mathbf{a})\mathbf{c} + (\text{div} \mathbf{c})\mathbf{a}, \quad (4.4.26)$$

$$\text{grad}(\mathbf{a} \cdot \mathbf{c}) = (\text{grad} \mathbf{a})^t \mathbf{c} + (\text{grad} \mathbf{c})^t \mathbf{a}, \quad (4.4.27)$$

can be used to deduce

$$\begin{aligned} \text{div} \left(-(\text{div} \mathbf{w})\pi_{\mathbf{Y}}I + \pi_{\mathbf{Y}} \text{grad} \mathbf{w}^t \right) &= -\text{div} \mathbf{w} \text{grad} \pi_{\mathbf{Y}} + \text{grad} \mathbf{w}^t \text{grad} \pi_{\mathbf{Y}} \\ &= \text{div} \left(-\text{grad} \pi_{\mathbf{Y}} \otimes \mathbf{w} + \text{grad} \pi_{\mathbf{Y}} \cdot \mathbf{w}I \right). \end{aligned} \quad (4.4.28)$$

The differential formulas written above can be used to rewrite the linear motion equation (4.4.21) in terms of other equivalent models. For instance, by using (4.4.28) Equation (4.4.21) leads to

$$\begin{aligned} \rho_Y \ddot{\mathbf{w}} + \rho_Y \left(\alpha \dot{\mathbf{w}} + \beta \|\mathbf{v}_Y\| \dot{\mathbf{w}} + \beta \frac{\mathbf{v}_Y \otimes \mathbf{v}_Y}{\|\mathbf{v}_Y\|} \dot{\mathbf{w}} \right) \\ = -\operatorname{div} \mathbf{w} \operatorname{grad} \pi_Y + \operatorname{grad} \mathbf{w}^t \operatorname{grad} \pi_Y + \operatorname{grad} \left(\frac{\rho_Y c_Y^2}{\gamma} \operatorname{div} \mathbf{w} \right) + \mathbf{d}_Y, \end{aligned}$$

or equivalently, in its conservative form,

$$\begin{aligned} \rho_Y \ddot{\mathbf{w}} + \rho_Y \left(\alpha \dot{\mathbf{w}} + \beta \|\mathbf{v}_Y\| \dot{\mathbf{w}} + \beta \frac{\mathbf{v}_Y \otimes \mathbf{v}_Y}{\|\mathbf{v}_Y\|} \dot{\mathbf{w}} \right) \\ = \operatorname{div} (-\operatorname{grad} \pi_Y \otimes \mathbf{w} + \operatorname{grad} \pi_Y \cdot \mathbf{w} I) + \operatorname{grad} \left(\frac{\rho_Y c_Y^2}{\gamma} \operatorname{div} \mathbf{w} \right) + \mathbf{d}_Y, \end{aligned}$$

where the body force contributions in the two equations above are given by

$$\mathbf{d}_Y(y, t) = \det F_Z(y, t) \mathbf{b}_X(\mathbf{Z}(y, t), t) - \mathbf{b}_Y(y, t). \quad (4.4.29)$$

4.4.1 Effects of the gravity field

Usually, the body force contribution in the linear models stated above are produced by the gravity force. Consequently, let us consider that $\mathbf{b}_X(x, t) = \rho_X(x, t) \mathbf{g}(x)$ and $\mathbf{b}_Y(x, t) = \rho_Y(y, t) \mathbf{g}(y)$ where $\mathbf{g}$ is the intensity of gravity field. Then, using (4.4.29) and the mass conservation equation for the motion $\mathbf{Y}$, we obtain

$$\begin{aligned} \mathbf{d}_Y(y, t) &= \det F_Z(y, t) \mathbf{b}_X(\mathbf{Z}(y, t), t) - \mathbf{b}_Y(y, t) \\ &= \det F_Z(y, t) \rho_X(\mathbf{Z}(y, t), t) \mathbf{g}(\mathbf{Z}(y, t)) - \rho_Y(y, t) \mathbf{g}(y) \\ &= \rho_Y(y, t) (\mathbf{g}(\mathbf{Z}(y, t)) - \mathbf{g}(y)) \\ &= \rho_Y(y, t) \operatorname{grad} \mathbf{g}(y) \mathbf{w}(y, t) + o(\mathbf{w}(y, t)). \end{aligned}$$

This approximation on the body forces and the linear motion equation (4.4.21) leads to the **Darcy-Forchheimer-Galbrun** linear model for small perturbations of a flow in an rigid-frame porous material under gravity,

$$\begin{aligned} \rho_Y \ddot{\mathbf{w}} + \rho_Y \left(\alpha \dot{\mathbf{w}} + \beta \|\mathbf{v}_Y\| \dot{\mathbf{w}} + \beta \frac{\mathbf{v}_Y \otimes \mathbf{v}_Y}{\|\mathbf{v}_Y\|} \dot{\mathbf{w}} \right) \\ = -\operatorname{div} \mathbf{w} \operatorname{grad} \pi_Y + \operatorname{grad} \mathbf{w}^t \operatorname{grad} \pi_Y + \operatorname{grad} \left(\frac{\rho_Y c_Y^2}{\gamma} \operatorname{div} \mathbf{w} \right) + \rho_Y (\operatorname{grad} \mathbf{g}) \mathbf{w}. \end{aligned}$$

However, for most of the acoustic applications, and in particular in this work, the variations of the gravity intensity are negligible due to the small scale of the physical domains of interest. Consequently, the body force term will be neglected.

4.5 Modelling small perturbations from an underlying incompressible fluid on a limited domain

A particular example of special interest consists in the wave propagation problems from an underlying flow produced by the motion $\mathbf{Y}$ of an incompressible fluid throughout the pore microstructure of the porous material with rigid solid frame. In that case, using standard arguments of continuum mechanics, the conservation of mass (4.4.5) for the motion $\mathbf{Y}$ leads to

$$\phi \dot{\rho}_{\mathbf{Y}}(y, t) + \rho_{\mathbf{Y}} \operatorname{div} \mathbf{v}_{\mathbf{Y}} = 0.$$

Now, taking into account the incompressibility of the motion $\mathbf{Y}$, this is, $\operatorname{div} \mathbf{v}_{\mathbf{Y}} = 0$ and so $\dot{\rho}_{\mathbf{Y}} = 0$, or in other words, the mass density is independent of time throughout the trajectory of motion $\mathbf{Y}$, taking into account that $\det F_{\mathbf{Y}} = 1$ (see [46, Chapter 57]):

$$\rho_{\mathbf{Y}}(y, t) = \phi \rho_0(\mathbf{P}_{\mathbf{Y}}(y, t)).$$

In many engineering applications in computational acoustics, the variations of the fluid mass density are negligible (except those applications related to oceanic, atmospheric and mantle convection where the spatial variations of the fluid mass density are important.). So, in this particular case, we can assume that the fluid mass density ρ_0 constant. Under this assumption, neglecting the spatial variations of the mass density and the gravity force, we have

$$\begin{aligned} \phi \rho_0 \ddot{\mathbf{w}} + \phi \rho_0 \left(\alpha \dot{\mathbf{w}} + \beta \|\mathbf{v}_{\mathbf{Y}}\| \dot{\mathbf{w}} + \beta \frac{\mathbf{v}_{\mathbf{Y}} \otimes \mathbf{v}_{\mathbf{Y}}}{\|\mathbf{v}_{\mathbf{Y}}\|} \dot{\mathbf{w}} \right) \\ = - \operatorname{div} \mathbf{w} \operatorname{grad} \pi_{\mathbf{Y}} + \operatorname{grad} \mathbf{w}^t \operatorname{grad} \pi_{\mathbf{Y}} + \operatorname{grad} \left(\frac{\phi \rho_0 c_{\mathbf{Y}}^2}{\gamma} \operatorname{div} \mathbf{w} \right), \end{aligned} \quad (4.5.1)$$

where the underlying velocity field satisfies $\operatorname{div} \mathbf{v}_{\mathbf{Y}} = 0$.

Other particular and simpler cases can be obtained from the motion equation written above, if $\beta = 0$, the tensor $\sigma = \phi \rho_0 \alpha$ is the well-known flow resistivity tensor in the porous Darcy-like models (see for instance [6, 20]). Finally, if $\phi = 1$ and $\alpha = \beta = 0$, then the isothermal version of the classical Galbrun's model for fluids is obtained [44, 54].

4.5.1 Dissipation of the energy

In the simplest case of parallel shear flows, $p_{\mathbf{Y}}$ and $\mathbf{v}_{\mathbf{Y}}$ are uniform, this is, they are given by a constant value. Hence, the linear governing equation (4.5.1) for the small perturbations from an underlying flow on a porous material is reduced to

$$\phi \rho_0 \ddot{\mathbf{w}} + \phi \rho_0 \left(\alpha \dot{\mathbf{w}} + \beta \|\mathbf{v}_{\mathbf{Y}}\| \dot{\mathbf{w}} + \beta \frac{\mathbf{v}_{\mathbf{Y}} \otimes \mathbf{v}_{\mathbf{Y}}}{\|\mathbf{v}_{\mathbf{Y}}\|} \dot{\mathbf{w}} \right) = \operatorname{grad} \left(\frac{\phi \rho_0 c_{\mathbf{Y}}^2}{\gamma} \operatorname{div} \mathbf{w} \right). \quad (4.5.2)$$

The right-hand term of the equation above is related with the Eulerian fluctuation of the pressure field π_E (in the trajectory of the motion $\mathbf{Y}$), more precisely,

$$\pi_E = -\frac{\phi\rho_0 c_Y^2}{\gamma} \operatorname{div} \mathbf{w} - \mathbf{w} \cdot \operatorname{grad} \pi_Y = -\frac{\phi\rho_0 c_Y^2}{\gamma} \operatorname{div} \mathbf{w}, \quad (4.5.3)$$

since the underlying flow is assumed uniform (π_Y is constant). This governing equation (4.5.2) can be used to show an energy conservation principle as follows.

Proposition 4.5.1. *Let $\mathbf{w}$ be the fluctuation displacement from an underlying uniform motion with constant velocity $\mathbf{v}_Y$ satisfying (4.5.2), then it holds*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\int_{\mathcal{P}_t^Y} \phi\rho_0 \|\dot{\mathbf{w}}\|^2 dV_y + \int_{\mathcal{P}_t^Y} \frac{\gamma}{\phi\rho_0 c_Y^2} |\pi_E|^2 dV_y \right) \\ & + \int_{\mathcal{P}_t^Y} \phi\rho_0 \left(\alpha \dot{\mathbf{w}} + \beta \|\mathbf{v}_Y\| \dot{\mathbf{w}} + \beta \frac{\mathbf{v}_Y \otimes \mathbf{v}_Y}{\|\mathbf{v}_Y\|} \dot{\mathbf{w}} \right) \cdot \dot{\mathbf{w}} dV_y = - \int_{\partial \mathcal{P}_t^Y} \pi_E \dot{\mathbf{w}} \cdot \boldsymbol{\nu} dA_y. \end{aligned} \quad (4.5.4)$$

Proof. First, Equation (4.5.2) is multiplied scalarly by $\dot{\mathbf{w}}$ and then it is integrated in the deformed configuration of the motion $\mathbf{Y}$. After using a Green's formula, we have

$$\begin{aligned} & \int_{\mathcal{P}_t^Y} \phi\rho_0 \dot{\mathbf{w}} \cdot \dot{\mathbf{w}} dV_y + \int_{\mathcal{P}_t^Y} \frac{\phi\rho_0 c_Y^2}{\gamma} \operatorname{div} \mathbf{w} \operatorname{div} \dot{\mathbf{w}} dV_y \\ & + \int_{\mathcal{P}_t^Y} \phi\rho_0 \left(\alpha \dot{\mathbf{w}} + \beta \|\mathbf{v}_Y\| \dot{\mathbf{w}} + \beta \frac{\mathbf{v}_Y \otimes \mathbf{v}_Y}{\|\mathbf{v}_Y\|} \dot{\mathbf{w}} \right) \cdot \dot{\mathbf{w}} dV_y \\ & = - \int_{\partial \mathcal{P}_t^Y} \frac{\phi\rho_0 c_Y^2}{\gamma} \operatorname{div} \mathbf{w} (\dot{\mathbf{w}} \cdot \boldsymbol{\nu}) dA_y. \end{aligned}$$

Now, taking into account (4.5.3) and the material time derivatives equities,

$$\dot{\mathbf{w}} \cdot \dot{\mathbf{w}} = \frac{1}{2} \overline{\|\dot{\mathbf{w}}\|^2}, \quad \frac{\phi\rho_0 c_Y^2}{\gamma} \operatorname{div} \mathbf{w} \operatorname{div} \dot{\mathbf{w}} = \frac{\gamma}{\phi\rho_0 c_Y^2} \pi_E \dot{\pi}_E = \frac{1}{2} \frac{\gamma}{\phi\rho_0 c_Y^2} \overline{|\pi_E|^2},$$

it holds

$$\begin{aligned} & \frac{1}{2} \int_{\mathcal{P}_t^Y} \phi\rho_0 \overline{\|\dot{\mathbf{w}}\|^2} dV_y + \frac{1}{2} \int_{\mathcal{P}_t^Y} \frac{\gamma}{\phi\rho_0 c_Y^2} \overline{|\pi_E|^2} dV_y \\ & + \int_{\mathcal{P}_t^Y} \phi\rho_0 \left(\alpha \dot{\mathbf{w}} + \beta \|\mathbf{v}_Y\| \dot{\mathbf{w}} + \beta \frac{\mathbf{v}_Y \otimes \mathbf{v}_Y}{\|\mathbf{v}_Y\|} \dot{\mathbf{w}} \right) \cdot \dot{\mathbf{w}} dV_y = - \int_{\partial \mathcal{P}_t^Y} \pi_E \dot{\mathbf{w}} \cdot \boldsymbol{\nu} dA_y. \end{aligned}$$

The result follows by using Prop. 4.3.1 with respect to the motion $\mathbf{Y}$. $\square$

From the proposition written above, a results on the dissipation of the energy can be stated by the following corollary:

Corollary 4.5.1. *Under the conditions of Prop. 4.5.1, if we define the energy (inertial and potential energy) associated with (4.5.2)*

$$E(t) = \int_{\mathcal{P}_t^{\mathbf{Y}}} \phi \rho_0 \|\dot{\mathbf{w}}(y, t)\|^2 dV_y + \int_{\mathcal{P}_t^{\mathbf{Y}}} \frac{\gamma}{\phi \rho_0 c_{\mathbf{Y}}^2} |\pi_{\mathbf{Y}}(y, t)|^2 dV_y,$$

then, if there does not exist any loads (external pressures) on $\partial \mathcal{P}_t^{\mathbf{Y}}$ and tensors α and β are semi-definite positive tensors then it holds

$$E(t) \leq E(t_0) \quad \text{for all } t \geq t_0.$$

Proof. First, notice the tensor $\mathbf{v}_{\mathbf{Y}} \otimes \mathbf{v}_{\mathbf{Y}} / \|\mathbf{v}_{\mathbf{Y}}\|$ is semi-definite positive (because its eigenvalues are 0 (double) and $\|\mathbf{v}_{\mathbf{Y}}\|$). In addition, if α and β are semi-definite positive tensors then all the terms in the second integral of the left-hand side of (4.5.4) are non negative. Finally, if no loads are considered on the boundary ($\pi_E = 0$), then the boundary term in (4.5.4) is null and hence the energy estimate is obtained since $E'(t) \leq 0$. $\square$

Chapter 5

Acoustic scattering by particle-velocity probes in presence of laminar flow

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5.1 Introduction

The wave propagation phenomena in compressible fluids is obviously affected by the particular values of the mass density and the sound speed. We have already in previous chapters that both physical quantities determine the governing equation for homogeneous compressible fluids at rest. However, it is usual at outdoor engineering applications that the fluid surrounding a structure is moving due for instance to the wind surrounding a mechanical structure of interest, or even by the relative motion of the structure with respect to a fluid which is steady and at rest. Hence, it is necessary to include the effects of the underlying flow in the acoustic fluid model.

For this purpose, there exists a variety of models such as the linearized Eulerian equations [10] or the Galbrun model (see the recent review [58] and reference therein). Since the latter model is written uniquely in terms of the acoustic perturbation of the displacement field, the Galbrun model have been chosen. Anyway, the first step to compute the numerical approximation of an aeroacoustic model consists in the computation of the underlying flow through fluid and porous materials, and then, using as input quantities those ones related with the flow, the time-harmonic coupled porous-fluid acoustic Galbrun model will be written.

5.2 Mathematical modelling

Since the acoustic scattering problem involves the propagation of time-harmonic waves in fluid and porous media under the presence of an underlying incompressible flow, three different mathematical models has to be described: the model that governs the underlying flow, the governing equations to model the acoustic perturbation of compressible fluids, and a similar model for porous materials with rigid solid frame.

5.2.1 Underlying flow model

The mathematical modelling of the underlying flow uses the most basic principles of the continuum mechanics. Firstly, we consider the conservation of mass (also called the continuity equation), which is given by

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \quad (5.2.1)$$

where t is the time variable, ρ is the Eulerian (spatial) description of the mass density and $\mathbf{v}$ is the Eulerian (spatial) description of the material velocity of the underlying motion.

Second, we have to taking into account the balance of forces (also called momentum equation), which is described by

$$\rho \frac{D\mathbf{v}}{Dt} + \rho \alpha \mathbf{v} + \rho \beta \|\mathbf{v}\| \mathbf{v} = \operatorname{div} T + \mathbf{b}, \quad (5.2.2)$$

where D/Dt is the material time derivative associated with the motion, this is

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial\mathbf{v}}{\partial t} + (\text{grad } \mathbf{v})\mathbf{v},$$

T is the Cauchy tensor, and $\mathbf{b}$ accounts for the body forces. Typically, the body forces are related with the gravity effects, but due to the nature of the fluids, these effects will be neglected in this work. The balance of forces law (5.2.2) is difference from the moments conservation equation since it contains the so-called Darcy (depending on tensor α) and the Forchheimer correction (depending on tensor β), which adds some dissipative behaviour on the model. Obviously, assuming those terms are null, we recover the classical momentum equation for fluids. Otherwise, if any of the tensors α and β are not null, then the governing equation models the motion of a porous material with rigid-solid frame [70].

Both conservation laws (5.2.1) and (5.2.2) are very general and valid for a wide class of materials. In this particular project, we discard the effects of the compressibility of the underlying flow (and hence also the noise produced by the its own motion at turbulent regime). Consequently, to simplify the computation of the underling motion, we will assume the fluid is incompressible and Newtonian. Both assumptions are described as follows. Since the fluid flow is incompressible, the mass density is constant, i.e., $\rho = \rho_0$ and from (5.2.1), we have

$$\text{div } \mathbf{v} = 0. \quad (5.2.3)$$

If the fluid is assumed Newtonian, the Cauchy stress tensor is given by the generalised form of Newton's law of viscosity:

$$T = - \left(p + \frac{2}{3}\mu \text{grad } \mathbf{v} \right) \mathbf{I} + \mu(\nabla\mathbf{v} + \nabla\mathbf{v}^T) \quad (5.2.4)$$

where p is the total pressure field and μ is the dynamic viscosity coefficient. Inserting the Newtonian constitutive law in the motion equation (5.2.2), we obtain

$$\rho_0 \frac{D\mathbf{v}}{Dt} + \rho\alpha\mathbf{v} + \rho\beta\|\mathbf{v}\|\mathbf{v} = -\text{grad } p + \mu\Delta\mathbf{v} + \frac{1}{3}\mu \text{grad div } \mathbf{v}. \quad (5.2.5)$$

To compute the underlying flow around a given mechanical structure (such as the PU probe), some boundary conditions should be added to the motion equation (5.2.5). More precisely, if Ω_S denotes the spatial domain where the rigid structure is placed, and Ω_P refers to the position of the porous material, then the fluid domain Ω_F will be considered as a bounding box, this is, $(a, A) \times (b, B) \times (c, C)$ surrounding that structure, such that $\Omega_S \subset (a, A) \times (b, B) \times (c, C)$, $\Omega_P \subset (a, A) \times (b, B) \times (c, C)$, and so the fluid domain will be given by $\Omega_F = ((a, A) \times (b, B) \times (c, C)) \setminus (\overline{\Omega_S} \cup \overline{\Omega_P})$.

Consequently, the boundary of the fluid domain Ω_F can be split in four disjoint parts: the boundary of the structure (obstacle) $\Gamma_S = \partial\Omega_S$, the coupling boundary with the porous subdomain, $\Gamma_C = \partial\Omega_P$, the inlet/outlet (front/back) boundary Γ_{in} and Γ_{out} , respectively, where the input and output velocity are imposed as known, and finally, the top/bottom

and left/right symmetric wall boundaries Γ_W where typically slip conditions are imposed for the fluid flow.

For simplicity, we will assume that the underlying flow is steady-state, this is, the pressure $p = p_0$ and the velocity field $\mathbf{v} = \mathbf{v}_0$ are time-independent. Moreover, since the porous material with rigid solid frame will be assumed homogeneous, we suppose that tensors α and β are constant.

Taking into account the motion equation, the boundary conditions described above and the steady-state assumption, then the computation of the pressure field $p_{F,0}$ and the velocity field $\mathbf{v}_{F,0}$ associated with the underling flow is written as follows: *Find steady-state fields $p_{F,0}$, $p_{P,0}$, $\mathbf{v}_{F,0}$, and $\mathbf{v}_{P,0}$ such that*

$$\rho_0 \frac{D\mathbf{v}_{F,0}}{Dt} = -\text{grad } p_{F,0} + \mu \Delta \mathbf{v}_{F,0} + \frac{\mu}{3} \text{grad div } \mathbf{v}_{F,0} \quad \text{in } \Omega_F, \quad (5.2.6)$$

$$\rho_0 \left(\frac{D\mathbf{v}_{P,0}}{Dt} + \alpha \mathbf{v}_{P,0} + \beta ||\mathbf{v}_{P,0}|| \mathbf{v}_{P,0} \right) = -\text{grad } p_{P,0} + \mu \Delta \mathbf{v}_{P,0} + \frac{\mu}{3} \text{grad div } \mathbf{v}_{P,0} \quad \text{in } \Omega_P, \quad (5.2.7)$$

$$\mathbf{v}_{F,0} = \mathbf{v}_{P,0} \quad \text{on } \Gamma_C, \quad (5.2.8)$$

$$p_{F,0} = p_{P,0} \quad \text{on } \Gamma_C, \quad (5.2.9)$$

$$\mathbf{v}_{F,0} = \mathbf{0} \quad \text{on } \Gamma_S, \quad (5.2.10)$$

$$\mathbf{v}_{F,0} = \mathbf{v}_{\text{in}}, \quad \frac{\partial p_{F,0}}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_{\text{in}}, \quad (5.2.11)$$

$$\mathbf{v}_{F,0} \cdot \boldsymbol{\tau} = 0, \quad p_{F,0} = 0 \quad \text{on } \Gamma_{\text{out}}, \quad (5.2.12)$$

$$\mathbf{v}_{F,0} \cdot \mathbf{n} = 0, \quad \frac{\partial p_{F,0}}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_W, \quad (5.2.13)$$

where $\mathbf{v}_{\text{in}}$ is the inlet velocity field (assume known data on the inlet boundary far from the obstacle), and $\mathbf{n}$ and $\boldsymbol{\tau}$ denote the unit normal and tangent vector on the boundaries of the fluid domain. The discretization method and the computational implementation details to solve of this underlying flow model is out of the scope of the present work and they are briefly discussed in Appendix .1. For the sake of simplicity on the notation and to ease the reading of the following sections, $\mathbf{v}_0$ and p_0 will denote the underlying velocity and pressure field in both fluid and porous subdomains, respectively.

5.2.2 Galbrun model for fluids and porous media

As it can be studied in detail in Chapter 4, the Galbrun model for fluid or porous material is the result of linearization of the mass conservation and the momentum equation using an Arbitrary Eulerian-Lagrangian description. Typically, the Galbrun model is obtained for fluid media assuming that the motion is isentropic and hence, the motion equation in the Galbrun model is given by

$$\rho_0 \frac{D^2 \mathbf{u}_F}{Dt^2} - \text{grad}(\rho_0 c_0^2 \text{div } \mathbf{u}_F) + \text{div } \mathbf{u}_F \text{grad } p_0 - (\text{grad } \mathbf{u}_F)^T \text{grad } p_0 = \mathbf{0} \quad \text{in } \Omega_F,$$

where D/Dt is the material time derivative associated with the underlying flow, this is,

$$\frac{D\mathbf{u}_F}{Dt} = \frac{\partial \mathbf{u}_F}{\partial t} + (\text{grad } \mathbf{v}_0) \mathbf{u}_F,$$

and recall $p_{F,0}$ and $\mathbf{v}_{F,0}$ are assumed as input known data (computed previously from the underlying flow problem (5.2.6)-(5.2.13)). At this point, it should be remarked that we are abusing on the notation because the differential operators grad, div and curl are used indistinctly for differentiation in Eulerian (see the section above) or arbitrary Lagrangian-Eulerian coordinates.

However, Ben-Dhia et al. [26] suggests this equation leads to “numerically polluted” result with typical piecewise polynomial Lagrange discretization due to $H^1(\Omega_F)$ coercivity issues. To avoid those drawbacks, the Galbrun model can be augmented with a “regularized” term proposed also in [26], where the vorticity field $\boldsymbol{\psi}_F = \text{curl } \mathbf{u}_F$ is introduced in the Galbrun model as follows:

$$\begin{aligned} \rho_0 \frac{D^2 \mathbf{u}_F}{Dt^2} - \text{grad}(\rho_0 c_0^2 \text{div } \mathbf{u}_F) + \text{curl}(\rho_0 c_0^2 (\text{curl } \mathbf{u}_F - \boldsymbol{\psi}_F)) \\ + \text{div } \mathbf{u}_F \text{grad } p_0 - (\text{grad } \mathbf{u}_F)^T \text{grad } p_0 = \mathbf{0} \quad \text{in } \Omega_F. \end{aligned}$$

In this manner, the augmented model is equivalent to the original one with an additional term, which should be null for the original solutions of the Galbrun model.

Since a new unknown field has been introduced, a motion equation for $\boldsymbol{\psi}$ must be taken into account. In this case, taking the curl on the original Galbrun model and doing some tedious algebraic computations, it holds

$$\frac{D^2 \boldsymbol{\psi}_F}{Dt^2} + 2 \frac{D}{Dt} (\mathcal{B}_F \mathbf{u}_F) + \mathcal{C}_F \mathbf{u}_F = \mathbf{0} \quad \text{in } \Omega_F, \quad (5.2.14)$$

where the differential operators $\mathcal{B}_F$ and $\mathcal{C}_F$ are given by

$$\mathcal{B}_F \mathbf{u}_F = \sum_{j=1}^3 \text{grad } v_{0,j} \wedge \frac{\partial \mathbf{u}_F}{\partial x_j}, \quad (5.2.15)$$

$$\begin{aligned} \mathcal{C}_F \mathbf{u}_F = & \sum_{j,k=1}^3 \left(\frac{\partial v_{0,k}}{\partial x_j} \text{grad } v_{0,j} \wedge \frac{\partial \mathbf{u}_F}{\partial x_k} - v_{0,j} \text{grad } \frac{\partial v_{0,k}}{\partial x_k} \wedge \frac{\partial \mathbf{u}_F}{\partial x_k} \right) \\ & + \frac{1}{\rho_0} \sum_{j=1}^3 \left(\frac{1}{\rho_0 c_0^2} \frac{\partial p_0}{\partial x_j} \text{grad } p_0 - \text{grad} \left(\frac{\partial p_0}{\partial x_j} \right) \right) \wedge \text{grad } u_{F,j}, \end{aligned} \quad (5.2.16)$$

being $\mathbf{v}_0 = \sum_{j=1}^3 v_{0,j} \mathbf{e}_j$ and analogously $\mathbf{u}_F = \sum_{j=1}^3 u_{F,j} \mathbf{e}_j$.

In the case of the Galbrun model for the porous materials with rigid solid frame, the Darcy-Forchheimer-Galbrun model derived in Chapter 4 is given by

$$\begin{aligned} \phi \rho_0 \frac{D^2 \mathbf{u}_P}{Dt^2} + \phi \rho_0 \left(\alpha \frac{D \mathbf{u}_P}{Dt} + \beta \|\mathbf{v}_0\| \frac{D \mathbf{u}_P}{Dt} + \beta \frac{\mathbf{v}_0 \otimes \mathbf{v}_0}{\|\mathbf{v}_0\|} \frac{D \mathbf{u}_P}{Dt} \right) \\ - \text{grad} \left(\frac{\phi \rho_0 c_0^2}{\gamma_0} \text{div } \mathbf{u}_P \right) + \text{div } \mathbf{u}_P \text{grad } p_0 - (\text{grad } \mathbf{u}_P)^T \text{grad } p_0 = \mathbf{0} \quad \text{in } \Omega_P, \end{aligned}$$

where recall ϕ is the porosity, α and β are tensors related to the Darcy and Forchheimer dissipative terms, and γ_0 is the ratio of specific heats at constant pressure and volume.

A similar augmented version could be derived for the Galbrun model for porous material using similar arguments, obtaining

$$\begin{aligned} \phi \rho_0 \frac{D^2 \mathbf{u}_P}{Dt^2} + \phi \rho_0 \left(\alpha \frac{D \mathbf{u}_P}{Dt} + \beta \|\mathbf{v}_0\| \frac{D \mathbf{u}_P}{Dt} + \beta \frac{\mathbf{v}_0 \otimes \mathbf{v}_0}{\|\mathbf{v}_0\|} \frac{D \mathbf{u}_P}{Dt} \right) \\ - \text{grad} \left(\frac{\phi \rho_0 c_0^2}{\gamma_0} \text{div } \mathbf{u}_P \right) + \text{curl} \left(\frac{\phi \rho_0 c_0^2}{\gamma_0} (\text{curl } \mathbf{u}_P - \boldsymbol{\psi}_P) \right) \\ + \text{div } \mathbf{u}_P \text{grad } p_0 - (\text{grad } \mathbf{u}_P)^T \text{grad } p_0 = \mathbf{0} \quad \text{in } \Omega_P, \end{aligned} \quad (5.2.17)$$

and

$$\frac{D^2 \boldsymbol{\psi}_P}{Dt^2} + 2 \frac{D}{Dt} (\mathcal{B}_P \mathbf{u}_P) + \mathcal{C}_P \mathbf{u}_P = \mathbf{0} \quad \text{in } \Omega_P, \quad (5.2.18)$$

where the operators $\mathcal{B}_P$ and $\mathcal{C}_P$ are defined by using analogous expressions to the fluid operators (5.2.15) and (5.2.16). However, to avoid an additional vector-valued unknown in the numerical computations, an identical low Mach number approximation will be performed both in the fluid and porous models.

5.2.3 Low-Mach number Approximation

For non-uniform laminar flow at low Mach numbers, with ρ_0 , p_0 , $\mathbf{v}_0$ and c_0 uniform, augmented Galbrun's equation is regularized (see [26]) in the time-harmonic domain as

follows:

$$\begin{aligned} \rho_0(-i\omega + \mathbf{v}_0 \cdot \text{grad})^2 \mathbf{u}_F - \text{grad}(\rho_0 c_0^2 \text{div } \mathbf{u}_F) + \text{curl}(\rho_0 c_0^2 (\text{curl } \mathbf{u}_F - \boldsymbol{\psi}_F)) \\ + \text{div } \mathbf{u}_F \text{ grad } p_0 - (\text{grad } \mathbf{u}_F)^T \cdot \text{grad } p_0 = \mathbf{0} \quad \text{in } \Omega_F, \end{aligned} \quad (5.2.19)$$

and

$$(-i\omega + \mathbf{v}_0 \cdot \text{grad})^2 \boldsymbol{\psi}_F + 2(-i\omega + \mathbf{v}_0 \cdot \text{grad})(\mathcal{B}_F \mathbf{u}) = \mathbf{0} \quad \text{in } \Omega_F, \quad (5.2.20)$$

In the low-Mach number approximation, the operator $\mathcal{C}_F$ is neglected and the operator. Hence, Equations (5.2.20) and (5.2.15) can be solved to obtain the following low Mach number approximation,

$$(\boldsymbol{\psi}_F)_{\text{LM}} = \frac{2}{i\omega} \mathcal{B}_F \mathbf{u}_F. \quad (5.2.21)$$

So, the motion equation for the augmented Galbrun model in a fluid medium with the low Mach number approximation reads as

$$\begin{aligned} \rho_0(-i\omega + \mathbf{v}_0 \cdot \text{grad})^2 \mathbf{u}_F - \text{grad}(\rho_0 c_0^2 \text{div } \mathbf{u}_F) + \text{curl} \left(\rho_0 c_0^2 (\text{curl } \mathbf{u}_F - \frac{2}{i\omega} \mathcal{B}_F \mathbf{u}_F) \right) \\ + \text{div } \mathbf{u}_F \text{ grad } p_0 - (\text{grad } \mathbf{u}_F)^t \text{ grad } p_0 = \mathbf{0}. \end{aligned} \quad (5.2.22)$$

Similarly, the time-harmonic motion equation of the augmented Darcy-Forchheimer-Galbrun model with the low Mach number approximation, which is assumed as

$$(\boldsymbol{\psi}_P)_{\text{LM}} = \frac{2}{i\omega} \mathcal{B}_P \mathbf{u}_P, \quad (5.2.23)$$

which yields to

$$\begin{aligned} \phi \rho_0(-i\omega + \mathbf{v}_0 \cdot \text{grad})^2 \mathbf{u}_P - \text{grad} \left(\frac{\phi \rho_0 c_0^2}{\gamma_0} \text{div } \mathbf{u}_P \right) \\ + \phi \rho_0 \alpha (-i\omega + \mathbf{v}_0 \cdot \text{grad}) \mathbf{u}_P + \phi \rho_0 \beta \|\mathbf{v}_0\| (-i\omega + \mathbf{v}_0 \cdot \text{grad}) \mathbf{u}_P \\ + \phi \rho_0 \beta \frac{\mathbf{v}_0 \otimes \mathbf{v}_0}{\|\mathbf{v}_0\|} (-i\omega + \mathbf{v}_0 \cdot \text{grad}) \mathbf{u}_P + \text{curl} \left(\frac{\phi \rho_0 c_0^2}{\gamma_0} (\text{curl } \mathbf{u}_P - \frac{2}{i\omega} \mathcal{B}_P \mathbf{u}_P) \right) \\ + \text{div } \mathbf{u}_P \text{ grad } p_0 - (\text{grad } \mathbf{u}_P)^T \text{ grad } p_0 = \mathbf{0} \quad \text{in } \Omega_P. \end{aligned} \quad (5.2.24)$$

Finally, notice that despite the governing equations at low-Mach number regime has been deduced assuming a uniform flow, they are still accurate approximations for non-uniform flows (see [41] for a detailed comparison with complex underlying flows). In fact, the error introduced by the low-Mach number approximation behaves like $O(|\mathbf{v}_0|/c_0)^2$.

5.3 Coupled scattering problem

Once we have been able to derive the motion equations which govern the classical Galbrun model and the Darcy-Forchheimer-Galbrun model at low-Mach number regime, we have all the ingredients to write a fluid-porous coupled model in the presence of underlying flow.

In the same manner as it has been written in the scattering problem at rest (without any flow) in Chapter 3, we assume that the computational domain is split in two disjoint subdomains, Ω_F and Ω_P , where the fluid and porous media are placed, respectively. The time-harmonic coupled scattering problem assume that the total displacement field in the fluid and porous subdomains are given by the addition of the scattered and the incident field, this is, $\mathbf{u}_F = \mathbf{u}^{\text{inc}} + \mathbf{u}_F^{\text{sc}}$ and $\mathbf{u}_P = \mathbf{u}^{\text{inc}} + \mathbf{u}_P^{\text{sc}}$, where typically $\mathbf{u}^{\text{inc}}$ is a planewave solution of the fluid Galbrun model at uniform flow.

To simplify the writing of the scattering problem, the definition of the coefficients α , β and ϕ will be piecewise extended such that tensors $\alpha = \beta = 0$ and $\phi = 1$ in the fluid subdomain. Consequently, taking into account this extension on the coefficients of the partial differential equations (5.2.17) and (5.2.18) will be the motion equation both for fluid and porous subdomains in the augmented formulation. Analogously, the displacement field $\mathbf{u}$ defined in the entire physical domain Ω will denote the fluid displacement field $\mathbf{u}_F$ in the fluid domain Ω_F and the porous displacement field $\mathbf{u}_P$ in the porous domain Ω_P . We will do the same considerations for the vorticity field $\boldsymbol{\psi}$ and the operators $\mathcal{B}$ and $\mathcal{C}$.

Additionally, for the sake of brevity in the description of the scattering problem, let us introduce $\mathcal{H}$ as the differential operator associated with (5.2.17), so this equation will be equivalent to $\mathcal{H}(\mathbf{u}, \boldsymbol{\psi}) = 0$.

Hence, the time-harmonic scattering problem in the presence of an underlying flow can be described as follows: *Given a fixed frequency $\omega > 0$ and an incident time-harmonic wave $\mathbf{u}^{\text{inc}}$, find $\mathbf{u} = \mathbf{u}^{\text{inc}} + \mathbf{u}^{\text{sc}}$ and $\boldsymbol{\psi}$ such that*

$$\mathcal{H}(\mathbf{u}^{\text{inc}} + \mathbf{u}^{\text{sc}}, \boldsymbol{\psi}) = \mathbf{0} \quad \text{in } \Omega, \quad (5.3.1)$$

$$\frac{D^2 \boldsymbol{\psi}}{Dt^2} + 2 \frac{D}{Dt} \mathcal{B}(\mathbf{u}^{\text{inc}} + \mathbf{u}^{\text{sc}}) + \mathcal{C}(\mathbf{u}^{\text{inc}} + \mathbf{u}^{\text{sc}}) = \mathbf{0} \quad \text{in } \Omega, \quad (5.3.2)$$

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_S, \quad (5.3.3)$$

$$\boldsymbol{\psi} - \text{curl } \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_S, \quad (5.3.4)$$

$$\text{radiation conditions} \quad \text{at } |\mathbf{x}| \rightarrow \infty, \quad (5.3.5)$$

where the time-harmonic material derivative is given by $D/Dt = -i\omega + \mathbf{v}_0 \cdot \text{grad}$ operator. In the case of the low-Mach number approximation, the coupled problem is written in an analogous way introducing the differential operator $\mathcal{H}_{\text{LM}}$ associated with the partial differential equation (5.2.24). Consequently, the low-Mach number scattering problem is written as follows: *Given a fixed frequency $\omega > 0$ and an incident time-harmonic wave $\mathbf{u}^{\text{inc}}$,*

find $\mathbf{u} = \mathbf{u}^{\text{inc}} + \mathbf{u}^{\text{sc}}$ such that

$$\mathcal{H}_{\text{LM}}(\mathbf{u}^{\text{inc}} + \mathbf{u}^{\text{sc}}, \psi) = \mathbf{0} \quad \text{in } \Omega, \quad (5.3.6)$$

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{\text{S}}, \quad (5.3.7)$$

$$\text{curl } \mathbf{u} - \frac{2}{i\omega} \mathcal{B}\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_{\text{S}}, \quad (5.3.8)$$

$$\text{radiation conditions} \quad \text{at } |\mathbf{x}| \rightarrow \infty. \quad (5.3.9)$$

In both scattering problems, the radiation conditions should guarantee that no incoming waves are being propagated from regions far from the obstacle. Due to the presence of an underlying (possible non-uniform) flow, the statement of those radiations conditions are far from been simple and it is a open research topic. An alternative to the local imposition of those radiations boundary conditions is the use of Perfectly Matched Layers (PML).

5.4 Perfectly Matched Layers

The procedure to derive the governing equations related to the PML layer is similar to the arguments already used for the acoustic models without flow. Basically, the spatial coordinates in the PML layers are stretched formally by a complex-valued change of variables. The Cartesian PML layers are placed on the subdomain

$$\Omega_{\text{PML}} = \prod_{j=1}^3 [-L_j^*, L_j^*] \setminus \prod_{j=1}^3 [-L_j, L_j].$$

We also consider the same singular absorbing profile along the rest of the chapters in Cartesian coordinates. Hence, the new complex-valued spatial coordinates are given by

$$\tilde{x}_j(x_j) = \begin{cases} x_j & \text{if } |x_j| \leq L_j, \\ x_j + i \frac{c_0}{\omega} \int_{L_j}^{|x_j|} \frac{1}{|L_j^* - s|} ds & \text{if } |x_j| \in (L_j, L_j^*]. \end{cases}$$

If the spatial derivatives of the complex-value change of variable is given by

$$\gamma_j(x_j) = \frac{\partial \tilde{x}_j}{\partial x_j} = 1 + i \frac{c_0}{\omega |L_j^* - x_j|},$$

then it is straightforward to compute the Jacobian tensor $H = \sum_{j=1}^3 \gamma_j \mathbf{e}_j \otimes \mathbf{e}_j$ and its determinant $J = \det H$.

Following the same procedure as in the previous chapters, the differential operators involved in the original augmented formulation of the Galbrun model are replaced formally

by identical operators differential operators but now depending on the variables $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$. Hence, the PML governing equation results

$$\rho_0(-i\omega + \mathbf{v}_0 \cdot \widetilde{\text{grad}})^2 \mathbf{u} - \widetilde{\text{grad}}(\rho_0 c_0^2 \widetilde{\text{div}} \mathbf{u}) + \widetilde{\text{curl}}(\rho_0 c_0^2 (\widetilde{\text{curl}} \mathbf{u} - \boldsymbol{\psi})) = \mathbf{0} \quad (5.4.1)$$

$$(-i\omega + \mathbf{v}_0 \cdot \widetilde{\text{grad}})^2 \boldsymbol{\psi} = \mathbf{0} \quad (5.4.2)$$

Taking into account some standard results of vector calculus, the differential operators $\widetilde{\text{grad}}$, $\widetilde{\text{div}}$, and $\widetilde{\text{curl}}$ can be rewritten in terms of H and J as follows:

$$\begin{aligned} & -\rho_0 \omega^2 \mathbf{u} - 2i\rho_0 \omega (\text{grad } \mathbf{u}) H^{-1} \mathbf{v}_0 \\ & + \rho_0 \text{grad}[(\text{grad } \mathbf{u}) H^{-1} \mathbf{v}_0] H^{-1} \mathbf{v}_0 - \rho_0 c_0^2 H^{-1} \text{grad}(H^{-1} : \text{grad } \mathbf{u}) \\ & + \rho_0 c_0^2 J^{-1} H \text{curl}(J^{-1} H H \text{curl } H \mathbf{u}) - \rho_0 c_0^2 J^{-1} H \text{curl}(H \boldsymbol{\psi}) = \mathbf{0} \quad \text{in } \Omega_{\text{PML}}, \end{aligned} \quad (5.4.3)$$

and the vorticity equation, given by

$$-\omega^2 \boldsymbol{\psi} - 2i\omega (\text{grad } \boldsymbol{\psi}) H^{-1} \mathbf{v}_0 + \text{grad}[(\text{grad } \boldsymbol{\psi}) H^{-1} \mathbf{v}_0] H^{-1} \mathbf{v}_0 = \mathbf{0} \quad \text{in } \Omega_{\text{PML}}. \quad (5.4.4)$$

Notice that only the PML governing equation has to be obtained for the classical fluid Galbrun model with uniform flow, since it is assume that the underlying flow is uniform (constant) far from the structure boundary Γ_S and also is assumed that the subdomain of the porous material is bounded and hence it has not to be truncated with PML layers. Consequently, p_0 is also constant and the Galbrun model has been simplified.

Similarly, the governing equation of the PML for the Galbrun's model can be obtained under the assumption of the low-Mach number laminar flow. In this particular case, the operator $\tilde{\mathcal{B}}$ is null and the replacement of the standard differential operators with their complex-valued stretched versions leads to

$$\rho_0(-i\omega + \mathbf{v}_0 \cdot \widetilde{\text{grad}})^2 \mathbf{u} - \widetilde{\text{grad}}(\rho_0 c_0^2 \text{div } \mathbf{u}) + \widetilde{\text{curl}}(\rho_0 c_0^2 \widetilde{\text{curl}} \mathbf{u}) = \mathbf{0}, \quad (5.4.5)$$

and writing this equation in terms of the Jacobian tensor H and its determinant J , we have

$$\begin{aligned} & -\rho_0 \omega^2 \mathbf{u} - 2i\rho_0 \omega (\text{grad } \mathbf{u}) H^{-1} \mathbf{v}_0 \\ & + \rho_0 \text{grad}[(\text{grad } \mathbf{u}) H^{-1} \mathbf{v}_0] H^{-1} \mathbf{v}_0 - \rho_0 c_0^2 H^{-1} \text{grad}(H^{-1} : \text{grad } \mathbf{u}) \\ & + \rho_0 c_0^2 J^{-1} H \text{curl}(J^{-1} H H \text{curl } H \mathbf{u}) = \mathbf{0} \quad \text{in } \Omega_{\text{PML}}. \end{aligned} \quad (5.4.6)$$

5.5 Variational formulation

Since we have introduced two different coupled scattering problems involving the augmented Galbrun model and another one which takes into account the low-Mach number approximation, we will also to derive the weak formulation of both motion equations.

5.5.1 Augmented Galbrun model

For parallel shear flows, with ρ_0 , p_0 , $\mathbf{v}_0$ and c_0 uniform, augmented Galbrun's equation in the time-harmonic domain becomes

$$\rho_0(-i\omega + \mathbf{v}_0 \cdot \nabla)^2 \mathbf{u} - \nabla(\rho_0 c_0^2 \operatorname{div} \mathbf{u}) + \operatorname{curl}(\rho_0 c_0^2 (\operatorname{curl} \mathbf{u} - \boldsymbol{\psi})) = \mathbf{0}, \quad (5.5.1a)$$

$$(-i\omega + \mathbf{v}_0 \cdot \nabla)^2 \boldsymbol{\psi} = \mathbf{0}. \quad (5.5.1b)$$

The combined variational form needs to be developed, where we solve for 6 scalar unknowns i.e., the components of $\mathbf{u}$ and $\boldsymbol{\psi}$. In what follows and to get shorter expressions we will use ∇ to denote the gradient operator $\operatorname{grad}(\cdot)$. With that purpose, Eq. (5.5.1a) is multiplied by a regular function $\mathbf{v}$ and Eq. (5.5.1b) is multiplied by a regular function $\mathbf{w}$. So, we have

$$\underbrace{\int_{\Omega} \rho_0(-i\omega + \mathbf{v}_0 \cdot \nabla)^2 \mathbf{u} \cdot \mathbf{v} dV}_{\text{Term T1}} - \underbrace{\int_{\Omega} \nabla(\rho_0 c_0^2 \operatorname{div} \mathbf{u}) \cdot \mathbf{v} dV}_{\text{Term T2}} + \underbrace{\int_{\Omega} \operatorname{curl}(\rho_0 c_0^2 (\operatorname{curl} \mathbf{u} - \boldsymbol{\psi})) \cdot \mathbf{v} dV}_{\text{Term T3}} + \underbrace{\int_{\Omega} (-i\omega + \mathbf{v}_0 \cdot \nabla)^2 \boldsymbol{\psi} \cdot \mathbf{w} dV}_{\text{Term T4}} = 0. \quad (5.5.2)$$

Now, let us expand this integral equality term-by-term and apply Green's identities to simplify higher derivatives. The Term **T1**,

$$\begin{aligned} \int_{\Omega} \rho_0(-i\omega + \mathbf{v}_0 \cdot \nabla)^2 \mathbf{u} \cdot \mathbf{v} dV &= \int_{\Omega} \rho_0 [-\omega^2 \mathbf{u} - 2i\omega \mathbf{v}_0 \cdot \nabla \mathbf{u} + \mathbf{v}_0 \cdot \nabla(\mathbf{v}_0 \cdot \nabla \mathbf{u})] \cdot \mathbf{v} dV \\ &= - \int_{\Omega} \rho_0 \omega^2 \mathbf{u} \cdot \mathbf{v} dV - \int_{\Omega} 2i\rho_0 \omega (\mathbf{v}_0 \cdot \nabla \mathbf{u}) \cdot \mathbf{v} dV + \underbrace{\int_{\Omega} \rho_0 [\mathbf{v}_0 \cdot \nabla(\mathbf{v}_0 \cdot \nabla \mathbf{u})] \cdot \mathbf{v} dV}_{\text{Term T1.1}}. \end{aligned}$$

If the Term T1.1 is also expanded using a Gauss Theorem, we obtain

$$\begin{aligned} \int_{\Omega} \rho_0(-i\omega + \mathbf{v}_0 \cdot \nabla)^2 \mathbf{u} \cdot \mathbf{v} dV &= - \int_{\Omega} \rho_0 \omega^2 \mathbf{u} \cdot \mathbf{v} dV - \int_{\Omega} 2i\rho_0 \omega (\mathbf{v}_0 \cdot \nabla \mathbf{u}) \cdot \mathbf{v} dV \\ &\quad - \int_{\Omega} \rho_0 [(\mathbf{v}_0 \cdot \nabla \mathbf{u}) \cdot \mathbf{v}] \operatorname{div} \mathbf{v}_0 dV - \int_{\Omega} \rho_0 (\mathbf{v}_0 \cdot \nabla \mathbf{u}) \cdot (\mathbf{v}_0 \cdot \nabla \mathbf{v}) dV \\ &\quad + \int_{\partial\Omega} \rho_0 [(\mathbf{v}_0 \cdot \nabla \mathbf{u}) \cdot \mathbf{v}] (\mathbf{v}_0 \cdot \mathbf{n}) dS. \end{aligned}$$

Using [39, Eq. (4) in Theorem 2.3.2], the Term **T2** leads to

$$\int_{\Omega} -\nabla(\rho_0 c_0^2 \operatorname{div} \mathbf{u}) \cdot \mathbf{v} dV = \int_{\Omega} \rho_0 c_0^2 \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} dV - \int_{\partial\Omega} \rho_0 c_0^2 \operatorname{div} \mathbf{u} (\mathbf{v} \cdot \mathbf{n}) dS,$$

and similarly Using [39, Eq. (6) in Theorem 2.3.2]], the Term **T3** yields to

$$\begin{aligned} \int_{\Omega} \rho_0 c_0^2 \operatorname{curl}(\operatorname{curl} \mathbf{u} - \boldsymbol{\psi}) \cdot \mathbf{v} \, dV &= \int_{\Omega} \rho_0 c_0^2 (\operatorname{curl} \mathbf{u} - \boldsymbol{\psi}) \cdot \operatorname{curl} \mathbf{v} \, dV \\ &\quad + \int_{\partial\Omega} \rho_0 c_0^2 [(\operatorname{curl} \mathbf{u} - \boldsymbol{\psi}) \times \operatorname{curl} \mathbf{v}] \cdot \mathbf{n} \, dS, \end{aligned}$$

and the term **T4** (similar to Term T1) can be rewriting as

$$\begin{aligned} \int_{\Omega} (-i\omega + \mathbf{v}_0 \cdot \nabla)^2 \boldsymbol{\psi} \cdot \mathbf{w} \, dV &= - \int_{\Omega} \omega^2 \boldsymbol{\psi} \cdot \mathbf{w} \, dV - \int_{\Omega} 2i\omega (\mathbf{v}_0 \cdot \nabla \boldsymbol{\psi}) \cdot \mathbf{w} \, dV \\ &\quad + \int_{\Omega} [\mathbf{v}_0 \cdot \nabla (\mathbf{v}_0 \cdot \nabla \boldsymbol{\psi})] \cdot \mathbf{w} \, dV. \end{aligned}$$

and proceeding in an analogous way as it has been done with the term T.1.1, we obtain

$$\begin{aligned} \int_{\Omega} (-i\omega + \mathbf{v}_0 \cdot \nabla)^2 \boldsymbol{\psi} \cdot \mathbf{w} \, dV &= - \int_{\Omega} \omega^2 \boldsymbol{\psi} \cdot \mathbf{w} \, dV - \int_{\Omega} 2i\omega (\mathbf{v}_0 \cdot \nabla \boldsymbol{\psi}) \cdot \mathbf{w} \, dV \\ &\quad - \int_{\Omega} [(\mathbf{v}_0 \cdot \nabla \boldsymbol{\psi}) \cdot \mathbf{w}] \operatorname{div} \mathbf{v}_0 \, dV - \int_{\Omega} (\mathbf{v}_0 \cdot \nabla \boldsymbol{\psi}) \cdot (\mathbf{v}_0 \cdot \nabla \mathbf{w}) \, dV \\ &\quad + \int_{\partial\Omega} [(\mathbf{v}_0 \cdot \nabla \boldsymbol{\psi}) \cdot \mathbf{w}] (\mathbf{v}_0 \cdot \mathbf{n}) \, dS. \end{aligned}$$

Collecting all the integral expressions above, the weak form of equations (5.5.1a) and (5.5.1b) is given by

$$\begin{aligned} a_F((\mathbf{u}, \boldsymbol{\psi}), (\mathbf{v}, \mathbf{w})) &= - \int_{\partial\Omega} \rho_0 [(\mathbf{v}_0 \cdot \nabla \mathbf{u}) \cdot \mathbf{v}] (\mathbf{v}_0 \cdot \mathbf{n}) \, dS + \int_{\partial\Omega} \rho_0 c_0^2 \operatorname{div} \mathbf{u} (\mathbf{v} \cdot \mathbf{n}) \, dS \\ &\quad - \int_{\partial\Omega} \rho_0 [(\operatorname{curl} \mathbf{u} - \boldsymbol{\psi}) \times \operatorname{curl} \mathbf{v}] \cdot \mathbf{n} \, dS - \int_{\partial\Omega} [(\mathbf{v}_0 \cdot \nabla \boldsymbol{\psi}) \cdot \mathbf{w}] (\mathbf{v}_0 \cdot \mathbf{n}) \, dS, \end{aligned} \quad (5.5.3)$$

where the bilinear form a_F is defined by

$$\begin{aligned} a_F((\mathbf{u}, \boldsymbol{\psi}), (\mathbf{v}, \mathbf{w})) &= - \int_{\Omega} \rho_0 \omega^2 \mathbf{u} \cdot \mathbf{v} \, dV - \int_{\Omega} 2i\rho_0 \omega (\mathbf{v}_0 \cdot \nabla \mathbf{u}) \cdot \mathbf{v} \, dV \\ &\quad - \int_{\Omega} \rho_0 [(\mathbf{v}_0 \cdot \nabla \mathbf{u}) \cdot \mathbf{v}] \operatorname{div} \mathbf{v}_0 \, dV - \int_{\Omega} \rho_0 (\mathbf{v}_0 \cdot \nabla \mathbf{u}) \cdot (\mathbf{v}_0 \cdot \nabla \mathbf{v}) \, dV \\ &\quad + \int_{\Omega} \rho_0 c_0^2 \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} \, dV + \int_{\Omega} \rho_0 c_0^2 \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{v} \, dV \\ &\quad - \int_{\Omega} \rho_0 c_0^2 \boldsymbol{\psi} \cdot \operatorname{curl} \mathbf{v} \, dV - \int_{\Omega} \omega^2 \boldsymbol{\psi} \cdot \mathbf{w} \, dV - \int_{\Omega} 2i\omega (\mathbf{v}_0 \cdot \nabla \boldsymbol{\psi}) \cdot \mathbf{w} \, dV \\ &\quad - \int_{\Omega} [(\mathbf{v}_0 \cdot \nabla \boldsymbol{\psi}) \cdot \mathbf{w}] \operatorname{div} \mathbf{v}_0 \, dV - \int_{\Omega} (\mathbf{v}_0 \cdot \nabla \boldsymbol{\psi}) \cdot (\mathbf{v}_0 \cdot \nabla \mathbf{w}) \, dV. \end{aligned} \quad (5.5.4)$$

Analogous computations leads to a similar bilinear form a_P for the porous materials, where the Darcy and Forchheimer terms should be taken into account.

5.5.2 Perfectly Matched Layers for the augmented model

The weak form of the PML governing equations is gotten from Eqs. (5.4.3) and (5.4.4). As it has been computed in the fluid subdomain, let us consider $\mathbf{v}, \mathbf{w}$ smooth test functions and multiply (5.4.3) by $J\mathbf{v}$ and (5.4.4) by $J\mathbf{w}$, where recall $J = \det H$. Hence, we have

$$\underbrace{- \int_{\Omega_{\text{PML}}} \omega^2 \boldsymbol{\psi} \cdot \mathbf{w} J dV}_{\text{Term S1}} - \underbrace{\int_{\Omega_{\text{PML}}} 2i\omega(\nabla \boldsymbol{\psi}) H^{-1} \mathbf{v}_0 \cdot \mathbf{w} J dV}_{\text{Term S2}} + \underbrace{\int_{\Omega_{\text{PML}}} \nabla[(\nabla \boldsymbol{\psi}) H^{-1} \mathbf{v}_0] H^{-1} \mathbf{v}_0 \cdot \mathbf{w} J dV}_{\text{Term S3}} = 0 \quad (5.5.5)$$

and also

$$\begin{aligned} & \underbrace{- \int_{\Omega_{\text{PML}}} \rho_0 \omega^2 \mathbf{u} \cdot \mathbf{v} J dV}_{\text{Term R1}} - \underbrace{\int_{\Omega_{\text{PML}}} 2i\rho_0 \omega(\nabla \mathbf{u}) H^{-1} \mathbf{v}_0 \cdot \mathbf{v} J dV}_{\text{Term R2}} \\ & + \underbrace{\int_{\Omega_{\text{PML}}} \rho_0 \nabla[(\nabla \mathbf{u}) H^{-1} \mathbf{v}_0] H^{-1} \mathbf{v}_0 \cdot \mathbf{v} J dV}_{\text{Term R3}} - \underbrace{\int_{\Omega_{\text{PML}}} \rho_0 c_0^2 H^{-1} \nabla(H^{-1} : \nabla \mathbf{u}) \cdot \mathbf{v} J dV}_{\text{Term R4}} \\ & + \underbrace{\int_{\Omega_{\text{PML}}} \rho_0 c_0^2 \frac{1}{J} H \operatorname{curl} \left(\frac{1}{J} H H \operatorname{curl} H \mathbf{u} \right) \cdot \mathbf{v} J dV}_{\text{Term R5}} - \underbrace{\int_{\Omega_{\text{PML}}} \rho_0 c_0^2 \frac{1}{J} H \operatorname{curl}(H \boldsymbol{\psi}) \cdot \mathbf{v} J dV}_{\text{Term R6}} = 0 \end{aligned} \quad (5.5.6)$$

Let us consider first the term in (5.5.5). Term **S1** and **S2** do not require any further manipulation. The Term **S3** can be rewritten using standard vector calculus formulas and after the application of a Gauss integral formula, we have

$$\begin{aligned} \int_{\Omega_{\text{PML}}} \nabla[(\nabla \boldsymbol{\psi}) H^{-1} \mathbf{v}_0] H^{-1} \mathbf{v}_0 \cdot \mathbf{w} J dV &= \int_{\Omega_{\text{PML}}} (J H^{-1} \mathbf{v}_0) \cdot (\nabla[(\nabla \boldsymbol{\psi}) H^{-1} \mathbf{v}_0])^t \mathbf{w} dV \\ &= - \int_{\Omega_{\text{PML}}} \operatorname{div}(J H^{-1} \mathbf{v}_0) (\nabla \boldsymbol{\psi} H^{-1} \mathbf{v}_0 \cdot \mathbf{w}) dV \\ &\quad - \int_{\Omega_{\text{PML}}} (J H^{-1} \mathbf{v}_0 \cdot \nabla \mathbf{w}) \cdot (\nabla \boldsymbol{\psi} H^{-1} \mathbf{v}_0) dV - \int_{\partial \Omega_{\text{PML}}} (\nabla \boldsymbol{\psi} H^{-1} \mathbf{v}_0 \cdot \mathbf{w}) (J H^{-1} \mathbf{v}_0 \cdot \mathbf{n}) dS. \end{aligned}$$

Again, Eq. (5.5.6) must be explored term-by-term by expanding the stretched differential operators and using Green's identities to simplify higher order spatial derivatives. More precisely, terms **R1** and **R2** do not require any further manipulation. The Term **R3** can

be rewritten using a Gauss integration formula, which leads to

$$\begin{aligned} \int_{\Omega_{\text{PML}}} \rho_0 \nabla[(\nabla \mathbf{u}) H^{-1} \mathbf{v}_0] H^{-1} \mathbf{v}_0 \cdot \mathbf{v} J dV &= - \int_{\Omega_{\text{PML}}} \rho_0 \operatorname{div}(J H^{-1} \mathbf{v}_0) [(\nabla \mathbf{u}) H^{-1} \mathbf{v}_0 \cdot \mathbf{v}] dV \\ &\quad - \int_{\Omega_{\text{PML}}} \rho_0 (J H^{-1} \mathbf{v}_0 \cdot \nabla \mathbf{v}) \cdot [(\nabla \mathbf{u}) H^{-1} \mathbf{v}_0] dV \\ &\quad + \int_{\partial \Omega_{\text{PML}}} \rho_0 [(\nabla \mathbf{u}) (H^{-1} \mathbf{v}_0) \cdot \mathbf{v}] (J H^{-1} \mathbf{v}_0 \cdot \mathbf{n}) dS. \end{aligned} \quad (5.5.7)$$

The Term **R4** only requires a transposition of tensor H when it is applied on a inner product, this is,

$$- \int_{\Omega_{\text{PML}}} \rho_0 c_0^2 H^{-1} \nabla(H^{-1} : \nabla \mathbf{u}) \cdot \mathbf{v} J dV = - \int_{\Omega_{\text{PML}}} \rho_0 c_0^2 \nabla(H^{-1} : \nabla \mathbf{u}) \cdot (J H^{-t} \mathbf{v}) dV.$$

Using Green's identity in the equation above, we obtain

$$\begin{aligned} - \int_{\Omega_{\text{PML}}} \rho_0 c_0^2 H^{-1} \nabla(H^{-1} : \nabla \mathbf{u}) \cdot \mathbf{v} J dV &= \int_{\Omega_{\text{PML}}} \rho_0 c_0^2 (H^{-1} : \nabla \mathbf{u}) \operatorname{div}(J H^{-t} \mathbf{v}) dV \\ &\quad - \int_{\partial \Omega_{\text{PML}}} \rho_0 c_0^2 (H^{-1} : \nabla \mathbf{u}) (J H^{-t} \mathbf{v} \cdot \mathbf{n}) dS. \end{aligned} \quad (5.5.8)$$

The Term **R5** can be manipulated similarly: first, computing the transpose of H , it holds

$$\begin{aligned} \int_{\Omega_{\text{PML}}} \rho_0 c_0^2 \frac{1}{J} H \operatorname{curl} \left(\frac{1}{J} H H \operatorname{curl} H \mathbf{u} \right) \cdot \mathbf{v} J dV \\ = \int_{\Omega_{\text{PML}}} \rho_0 c_0^2 \operatorname{curl} \left(\frac{1}{J} H H \operatorname{curl} H \mathbf{u} \right) \cdot (H^t \mathbf{v}) dV, \end{aligned}$$

and then use the Green's identity (see [39, Eq. (6), Theorem 2.3.2]), it holds

$$\begin{aligned} = \int_{\Omega_{\text{PML}}} \rho_0 c_0^2 \left(\frac{1}{J} H H \operatorname{curl} H \mathbf{u} \right) \cdot \operatorname{curl}(H^t \mathbf{v}) dV \\ + \int_{\partial \Omega_{\text{PML}}} \rho_0 c_0^2 \left(\frac{1}{J} H H \operatorname{curl} H \mathbf{u} \times H^t \mathbf{v} \right) \cdot \mathbf{n} dS. \end{aligned} \quad (5.5.9)$$

Proceeding in an exact manner, term **R6** is rewritten as follows:

$$\begin{aligned} - \int_{\Omega_{\text{PML}}} \rho_0 c_0^2 \frac{1}{J} H^t \operatorname{curl}(H \boldsymbol{\psi}) \cdot \mathbf{v} J dV &= - \int_{\Omega_{\text{PML}}} \rho_0 c_0^2 \operatorname{curl}(H \boldsymbol{\psi}) \cdot H \mathbf{v} dV \\ &= - \int_{\Omega_{\text{PML}}} \rho_0 c_0^2 H \boldsymbol{\psi} \cdot \operatorname{curl} H \mathbf{v} dV - \int_{\partial \Omega_{\text{PML}}} \rho_0 c_0^2 (H \boldsymbol{\psi} \times H \mathbf{v}) \cdot \mathbf{n} dS. \end{aligned} \quad (5.5.10)$$

Collecting the expressions of all the terms **S1-S3** and **R1-R6**, we obtain the weak form associated with the PML model:

$$\begin{aligned}
a_{\text{PML}}((\mathbf{u}, \psi), (\mathbf{v}, \mathbf{w})) = & - \int_{\partial\Omega_{\text{PML}}} \rho_0 [(\nabla \mathbf{u})(H^{-1} \mathbf{v}_0) \cdot \mathbf{v}] (JH^{-1} \mathbf{v}_0 \cdot \mathbf{n}) dS \\
& + \int_{\partial\Omega_{\text{PML}}} \rho_0 c_0^2 (H^{-1} : \nabla \mathbf{u}) (JH^{-1} \mathbf{v} \cdot \mathbf{n}) dS \\
& - \int_{\partial\Omega_{\text{PML}}} \rho_0 c_0^2 \left(\frac{1}{J} H H \operatorname{curl} H \mathbf{u} \times H \mathbf{v} \right) \cdot \mathbf{n} dS \\
& + \int_{\partial\Omega_{\text{PML}}} \rho_0 c_0^2 (H \psi \times H \mathbf{v}) \cdot \mathbf{n} dS \\
& + \int_{\partial\Omega_{\text{PML}}} (\nabla \psi H^{-1} \mathbf{v}_0 \cdot \mathbf{w}) (JH^{-1} \mathbf{v}_0 \cdot \mathbf{n}) dS, \quad (5.5.11)
\end{aligned}$$

where the bilinear form a_{PML} is defined by

$$\begin{aligned}
a_{\text{PML}}((\mathbf{u}, \psi), (\mathbf{v}, \mathbf{w})) = & - \int_{\Omega_{\text{PML}}} \rho_0 \omega^2 J \mathbf{u} \cdot \mathbf{v} dV - \int_{\Omega_{\text{PML}}} 2i\rho_0 \omega J (\nabla \mathbf{u}) H^{-1} \mathbf{v}_0 \cdot \mathbf{v} dV \\
& - \int_{\Omega_{\text{PML}}} \rho_0 \operatorname{div}(JH^{-1} \mathbf{v}_0) [(\nabla \mathbf{u}) H^{-1} \mathbf{v}_0 \cdot \mathbf{v}] dV \\
& - \int_{\Omega_{\text{PML}}} \rho_0 (JH^{-1} \mathbf{v}_0 \cdot \nabla \mathbf{v}) \cdot [(\nabla \mathbf{u}) H^{-1} \mathbf{v}_0] dV \\
& + \int_{\Omega_{\text{PML}}} \rho_0 c_0^2 (H^{-1} : \nabla \mathbf{u}) \operatorname{div}(JH^{-1} \mathbf{v}) dV \\
& + \int_{\Omega_{\text{PML}}} \rho_0 c_0^2 \left(\frac{1}{J} H H \operatorname{curl} H \mathbf{u} \right) \cdot \operatorname{curl}(H \mathbf{v}) dV \\
& - \int_{\Omega_{\text{PML}}} \rho_0 c_0^2 H \psi \cdot \operatorname{curl} H \mathbf{v} dV - \int_{\Omega_{\text{PML}}} \omega^2 J \psi \cdot \mathbf{w} dV \\
& - \int_{\Omega_{\text{PML}}} 2i\omega J (\nabla \psi) H^{-1} \mathbf{v}_0 \cdot \mathbf{w} dV \\
& - \int_{\Omega_{\text{PML}}} \operatorname{div}(JH^{-1} \mathbf{v}_0) (\nabla \psi H^{-1} \mathbf{v}_0 \cdot \mathbf{w}) dV \\
& - \int_{\Omega_{\text{PML}}} (JH^{-1} \mathbf{v}_0 \cdot \nabla \mathbf{w}) \cdot (\nabla \psi H^{-1} \mathbf{v}_0) dV. \quad (5.5.12)
\end{aligned}$$

5.5.3 Low-Mach number approximation

If the low-Mach number approximation is assumed in the Galbrun model, the variational form only needs to take into account 3 unknowns i.e., the three components of the displacement field $\mathbf{u}$. The weak form of the Equation (5.2.22) can be written as usual,

multiplying this equation for a sooth test function $\mathbf{v}$, we obtain

$$\begin{aligned}
 & \underbrace{\int_{\Omega} \rho_0 (-i\omega + \mathbf{v}_0 \cdot \nabla)^2 \mathbf{u} \cdot \mathbf{v} dV}_{\text{Term L1}} - \underbrace{\int_{\Omega} \nabla(\rho_0 c_0^2 \operatorname{div} \mathbf{u}) \cdot \mathbf{v} dV}_{\text{Term L2}} \\
 & \quad + \underbrace{\int_{\Omega} \operatorname{curl}(\rho_0 c_0^2 (\operatorname{curl} \mathbf{u} - \frac{2}{i\omega} \mathcal{B}\mathbf{u})) \cdot \mathbf{v} dV}_{\text{Term L3}} \\
 & \quad + \underbrace{\int_{\Omega} \operatorname{div} \mathbf{u} \nabla p_0 \cdot \mathbf{v} dV}_{\text{Term L4}} - \underbrace{\int_{\Omega} (\nabla \mathbf{u}^t \cdot \nabla p_0) \cdot \mathbf{v} dV}_{\text{Term L5}} = 0. \tag{5.5.13}
 \end{aligned}$$

The weak form can be handled term-by-term as follows: Term **L1** requires the same kind of computations to those ones applied to Term **T1**, so we have

$$\begin{aligned}
 \int_{\Omega} \rho_0 (-i\omega + \mathbf{v}_0 \cdot \nabla)^2 \mathbf{u} \cdot \mathbf{v} dV &= - \int_{\Omega} \rho_0 \omega^2 \mathbf{u} \cdot \mathbf{v} dV \\
 &\quad - \int_{\Omega} 2i\rho_0 \omega (\mathbf{v}_0 \cdot \nabla \mathbf{u}) \cdot \mathbf{v} dV - \int_{\Omega} \rho_0 [(\mathbf{v}_0 \cdot \nabla \mathbf{u}) \cdot \mathbf{v}] \operatorname{div} \mathbf{v}_0 dV \\
 &\quad - \int_{\Omega} \rho_0 (\mathbf{v}_0 \cdot \nabla \mathbf{u}) \cdot (\mathbf{v}_0 \cdot \nabla \mathbf{v}) dV + \int_{\partial\Omega} \rho_0 [(\mathbf{v}_0 \cdot \nabla \mathbf{u}) \cdot \mathbf{v}] (\mathbf{v}_0 \cdot \mathbf{n}) dS.
 \end{aligned}$$

Term **L2**, which is similar to Term **T2**, leads to

$$\int_{\Omega} -\nabla(\rho_0 c_0^2 \operatorname{div} \mathbf{u}) \cdot \mathbf{v} dV = \int_{\Omega} \rho_0 c_0^2 \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} dV - \int_{\partial\Omega} \rho_0 c_0^2 \operatorname{div} \mathbf{u} (\mathbf{v} \cdot \mathbf{n}) dS,$$

and Term **L3** (again similar to Term **T3**) can be expressed by

$$\begin{aligned}
 & \int_{\Omega} \rho_0 c_0^2 \operatorname{curl}(\operatorname{curl} \mathbf{u} - \frac{2}{i\omega} \mathcal{B}\mathbf{u}) \cdot \mathbf{v} dV \\
 &= \int_{\Omega} \rho_0 c_0^2 (\operatorname{curl} \mathbf{u} - \frac{2}{i\omega} \mathcal{B}\mathbf{u}) \cdot \operatorname{curl} \mathbf{v} dV + \int_{\partial\Omega} \rho_0 c_0^2 \left[(\operatorname{curl} \mathbf{u} - \frac{2}{i\omega} \mathcal{B}\mathbf{u}) \times \mathbf{v} \right] \cdot \mathbf{n} dS.
 \end{aligned}$$

Terms **L4** and **L5** do not require any additional manipulation and they are used as is. Finally, collecting all the expressions written above, the final weak form is given by

$$\begin{aligned}
 a_F^{\text{LM}}(\mathbf{u}, \mathbf{v}) &= - \int_{\partial\Omega} \rho_0 [(\mathbf{v}_0 \cdot \nabla \mathbf{u}) \cdot \mathbf{v}] (\mathbf{v}_0 \cdot \mathbf{n}) dS + \int_{\partial\Omega} \rho_0 c_0^2 \operatorname{div} \mathbf{u} (\mathbf{v} \cdot \mathbf{n}) dS \\
 &\quad - \int_{\partial\Omega} \rho_0 c_0^2 \left[(\operatorname{curl} \mathbf{u} - \frac{2}{i\omega} \mathcal{B}\mathbf{u}) \times \operatorname{curl} \mathbf{v} \right] \cdot \mathbf{n} dS, \tag{5.5.14}
 \end{aligned}$$

where the bilinear form a_F^{LM} is defined by

$$\begin{aligned}
a_F^{\text{LM}}(\mathbf{u}, \mathbf{v}) = & - \int_{\Omega} \rho_0 \omega^2 \mathbf{u} \cdot \mathbf{v} dV - \int_{\Omega} 2i\rho_0 \omega (\mathbf{v}_0 \cdot \nabla \mathbf{u}) \cdot \mathbf{v} dV \\
& - \int_{\Omega} \rho_0 [(\mathbf{v}_0 \cdot \nabla \mathbf{u}) \cdot \mathbf{v}] \operatorname{div} \mathbf{v}_0 dV - \int_{\Omega} \rho_0 (\mathbf{v}_0 \cdot \nabla \mathbf{u}) \cdot (\mathbf{v}_0 \cdot \nabla \mathbf{v}) dV \\
& + \int_{\Omega} \rho_0 c_0^2 \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} dV + \int_{\Omega} \rho_0 c_0^2 (\operatorname{curl} \mathbf{u} - \frac{2}{i\omega} \mathcal{B} \mathbf{u}) \cdot \operatorname{curl} \mathbf{v} dV \\
& + \int_{\Omega} \operatorname{div} \mathbf{u} \nabla p_0 \cdot \mathbf{v} dV - \int_{\Omega} (\nabla \mathbf{u}^t \cdot \nabla p_0) \cdot \mathbf{v} dV. \quad (5.5.15)
\end{aligned}$$

Analogous computations leads to a similar bilinear form a_P^{LM} for the porous materials, where the Darcy and Forchheimer terms should be taken into account.

5.5.4 Perfectly Matched Layers in the low-Mach number approximation

Since Eq. (5.4.6) is similar to Eq. (5.4.3) with some additional simplification (coming from the fact that the underlying flow is assumed uniform inside the PML subdomain), then the terms and the variational form may be proceeded with accordingly.

Consequently, the form associated to the PML governing equation assuming a low-Mach number approximation is given by

$$\begin{aligned}
a_{\text{PML}}^{\text{LM}}(\mathbf{u}, \mathbf{v}) = & - \int_{\partial\Omega_{\text{PML}}} \rho_0 [(\nabla \mathbf{u})(H^{-1} \mathbf{v}_0) \cdot \mathbf{v}] (JH^{-1} \mathbf{v}_0 \cdot \mathbf{n}) dS \\
& + \int_{\partial\Omega_{\text{PML}}} \rho_0 c_0^2 (H^{-1} : \nabla \mathbf{u}) (JH^{-1} \mathbf{v} \cdot \mathbf{n}) dS \\
& - \int_{\partial\Omega_{\text{PML}}} \rho_0 c_0^2 \left(\frac{1}{J} H H \operatorname{curl} H \mathbf{u} \times H \mathbf{v} \right) \cdot \mathbf{n} dS,
\end{aligned}$$

where the bilinear form $a_{\text{PML}}^{\text{LM}}$ is given by

$$\begin{aligned}
 a_{\text{PML}}^{\text{LM}}(\mathbf{u}, \mathbf{v}) = & - \int_{\Omega_{\text{PML}}} \rho_0 \omega^2 J \mathbf{u} \cdot \mathbf{v} dV - \int_{\Omega_{\text{PML}}} 2i\rho_0 \omega J (\nabla \mathbf{u}) H^{-1} \mathbf{v}_0 \cdot \mathbf{v} dV \\
 & - \int_{\Omega_{\text{PML}}} \rho_0 \operatorname{div}(JH^{-1} \mathbf{v}_0) [(\nabla \mathbf{u}) H^{-1} \mathbf{v}_0 \cdot \mathbf{v}] dV \\
 & - \int_{\Omega_{\text{PML}}} \rho_0 (JH^{-1} \mathbf{v}_0 \cdot \nabla \mathbf{v}) \cdot [(\nabla \mathbf{u}) H^{-1} \mathbf{v}_0] dV \\
 & + \int_{\Omega_{\text{PML}}} \rho_0 c_0^2 (H^{-1} : \nabla \mathbf{u}) \operatorname{div}(JH^{-1} \mathbf{v}) dV \\
 & + \int_{\Omega_{\text{PML}}} \rho_0 c_0^2 \left(\frac{1}{J} H H \operatorname{curl} H \mathbf{u} \right) \cdot \operatorname{curl} H \mathbf{v} dV.
 \end{aligned}$$

5.6 Coupled weak scattering problems

Once we have derived the weak formulation associated with the governing equations of the augmented Galbrun model and its related PML equation, it is straightforward to write the weak form of the scattering problems described in the previous sections.

Firstly, since the scattered displacement field satisfies $\mathbf{u} = \mathbf{u}^{\text{inc}} + \mathbf{u}^{\text{sc}}$, let us introduce an adequate functional space to seek the scattered displacement field $\mathbf{u}^{\text{sc}}$ and the vorticity $\boldsymbol{\psi}$. If we take into account the regularity of the terms in the bilinear forms a_{F} , a_{P} , and a_{PML} , it must to impose that $\mathbf{u}^{\text{sc}}, \boldsymbol{\psi} \in \mathbf{V}$ where

$$\mathbf{V} = \{ \mathbf{v} \in [\mathbf{H}^1(\Omega)]^3 : H^{-1} \nabla \mathbf{u} \in [\mathbf{L}^2(\Omega)]^{3 \times 3}, J \mathbf{v} \in [\mathbf{H}^1(\Omega)]^3, \mathbf{v} = \mathbf{0} \text{ on } \Gamma^* \}.$$

Notice that in the definition of the functional space, we are abusing on the notation and tensor H and scalar J are extended to the entire computational domain using respectively the identity tensor and the unit constant in fluid and porous subdomains.

In the case of the augmented formulation, it is stated as follows: *Given a fixed frequency $\omega > 0$ and an incident time-harmonic wave $\mathbf{u}^{\text{inc}}$, find $\mathbf{u}^{\text{sc}}, \boldsymbol{\psi} \in \mathbf{V}$ satisfying $\mathbf{u}^{\text{sc}} \cdot \mathbf{n} = -\mathbf{u}^{\text{inc}} \cdot \mathbf{n}$ on Γ_{S} and such that*

$$\begin{aligned}
 a_{\text{F}}((\mathbf{u}^{\text{sc}}, \boldsymbol{\psi}), (\mathbf{v}, \mathbf{w})) + a_{\text{P}}((\mathbf{u}^{\text{sc}}, \boldsymbol{\psi}), (\mathbf{v}, \mathbf{w})) + a_{\text{PML}}((\mathbf{u}^{\text{sc}}, \boldsymbol{\psi}), (\mathbf{v}, \mathbf{w})) \\
 = -a_{\text{F}}((\mathbf{u}^{\text{inc}}, \mathbf{0}), (\mathbf{v}, \mathbf{w})) - a_{\text{P}}((\mathbf{u}^{\text{inc}}, \mathbf{0}), (\mathbf{v}, \mathbf{w})), \quad (5.6.1)
 \end{aligned}$$

for all test functions $\mathbf{v}, \mathbf{w} \in \mathbf{V}$ satisfying $\mathbf{v} \cdot \mathbf{n} = 0$ on Γ_{S} .

In the case of the low-Mach number approximation, the coupled problem is written in an analogous way taking into account the expressions of the bilinear forms a_{F}^{LM} , a_{P}^{LM} , and $a_{\text{PML}}^{\text{LM}}$. Those bilinear forms have similar differential operators and hence, we seek the

solution in the same functional space. Hence, the weak problem is described as follows: *Given a fixed frequency $\omega > 0$ and an incident time-harmonic wave $\mathbf{u}^{\text{inc}}$, find $\mathbf{u}^{\text{sc}} \in V$ satisfying $\mathbf{u}^{\text{sc}} \cdot \mathbf{n} = -\mathbf{u}^{\text{inc}} \cdot \mathbf{n}$ on Γ_S and such that*

$$a_F^{\text{LM}}(\mathbf{u}^{\text{sc}}, \mathbf{v}) + a_P^{\text{LM}}(\mathbf{u}^{\text{sc}}, \mathbf{v}) + a_{\text{PML}}^{\text{LM}}(\mathbf{u}^{\text{sc}}, \mathbf{v}) = -a_F^{\text{LM}}(\mathbf{u}^{\text{inc}}, \mathbf{v}) - a_P^{\text{LM}}(\mathbf{u}^{\text{inc}}, \mathbf{v}), \quad (5.6.2)$$

for all test functions $\mathbf{v}, \mathbf{w} \in V$ satisfying $\mathbf{v} \cdot \mathbf{n} = 0$ on Γ_S .

5.6.1 Finite element discretization

The numerical approximation of the two weak coupled problems stated above do not require any special treatment from a discretization point of view. In fact, as a previous step the underlying flow data, p_0 and $\mathbf{v}_0$ must be interpolated on a given simplicial mesh $\mathcal{T}_h$ of Ω , which is conformal with all the boundaries and coupling interfaces of the coupled problem. This interpolation procedure is computed by using a standard pointwise Lagrange interpolation to obtain the discrete fields $p_{0,h}$ and $\mathbf{v}_{0,h}$. Once the original continuous fields p_0 and $\mathbf{v}_0$ are replaced by $p_{0,h}$ and $\mathbf{v}_{0,h}$, it is possible to define the bilinear forms $a_{F,h}$, $a_{P,h}$, and $a_{\text{PML},h}$.

Identical Lagrange interpolation is used to compute the discrete field $\mathbf{u}_h^{\text{inc}}$ associated with the incident displacement field $\mathbf{u}^{\text{inc}}$. Finally, the functional space V is substituted by its finite-dimensional subspace

$$V_h = \{\mathbf{v} \in V : \mathbf{v}|_T \in [\mathbb{P}^1(\mathbb{C})]^3, T \in \mathcal{T}_h, \mathbf{v} = \mathbf{0} \text{ on } \Gamma^\star\},$$

using standard piecewise linear Lagrange finite elements. In the case of the augmented formulation, the discrete variational problem is stated as follows: *Given a fixed frequency $\omega > 0$ and an incident time-harmonic discrete wave $\mathbf{u}_h^{\text{inc}}$, find $\mathbf{u}_h^{\text{sc}}, \boldsymbol{\psi}_h \in V_h$ satisfying $\mathbf{u}_h^{\text{sc}} \cdot \mathbf{n} = -\mathbf{u}_h^{\text{inc}} \cdot \mathbf{n}$ on Γ_S and such that*

$$a_F^h((\mathbf{u}_h^{\text{sc}}, \boldsymbol{\psi}_h), (\mathbf{v}_h, \mathbf{w}_h)) + a_P^h((\mathbf{u}_h^{\text{sc}}, \boldsymbol{\psi}_h), (\mathbf{v}_h, \mathbf{w}_h)) + a_{\text{PML}}^h((\mathbf{u}_h^{\text{sc}}, \boldsymbol{\psi}_h), (\mathbf{v}_h, \mathbf{w}_h)) = -a_F((\mathbf{u}_h^{\text{inc}}, \mathbf{0}), (\mathbf{v}_h, \mathbf{w}_h)) - a_P((\mathbf{u}_h^{\text{inc}}, \mathbf{0}), (\mathbf{v}_h, \mathbf{w}_h)), \quad (5.6.3)$$

for all test functions $\mathbf{v}_h, \mathbf{w}_h \in V_h$ satisfying $\mathbf{v}_h \cdot \mathbf{n} = 0$ on Γ_S .

In the case of the low-Mach number approximation, the coupled problem is written in an analogous way taking into account the expressions of the bilinear forms $a_F^{\text{LM},h}$, $a_P^{\text{LM},h}$, and $a_{\text{PML}}^{\text{LM},h}$. Those discrete bilinear forms have similar differential operators and hence, we seek the solution in the same discrete finite element space. Hence, the discrete weak problem is described as follows: *Given a fixed frequency $\omega > 0$ and an incident time-harmonic wave $\mathbf{u}_h^{\text{inc}}$, find $\mathbf{u}_h^{\text{sc}} \in V_h$ satisfying $\mathbf{u}_h^{\text{sc}} \cdot \mathbf{n} = -\mathbf{u}_h^{\text{inc}} \cdot \mathbf{n}$ on Γ_S and such that*

$$a_F^{\text{LM},h}(\mathbf{u}_h^{\text{sc}}, \mathbf{v}_h) + a_P^{\text{LM},h}(\mathbf{u}_h^{\text{sc}}, \mathbf{v}_h) + a_{\text{PML}}^{\text{LM},h}(\mathbf{u}_h^{\text{sc}}, \mathbf{v}_h) = -a_F^{\text{LM},h}(\mathbf{u}_h^{\text{inc}}, \mathbf{v}_h) - a_P^{\text{LM},h}(\mathbf{u}_h^{\text{inc}}, \mathbf{v}_h), \quad (5.6.4)$$

for all test functions $\mathbf{v}_h \in V_h$ satisfying $\mathbf{v}_h \cdot \mathbf{n} = 0$ on Γ_S .

5.7 Validation tests

The finite element implementation of the discrete problems described above has been validated in a variety of benchmark tests with incremental complexity. First, the classical Galbrun model has been tested in a bounded fluid subdomain. Then, the classical duct test has been used to validate the PML implementation in a benchmark with known exact solution. Finally, a Galbrun model with non-uniform underlying flow is considered. In this latter case, the numerical results have been compared qualitatively with respect to the numerical results reported in [41].

5.7.1 Uniform-flow Galbrun equation

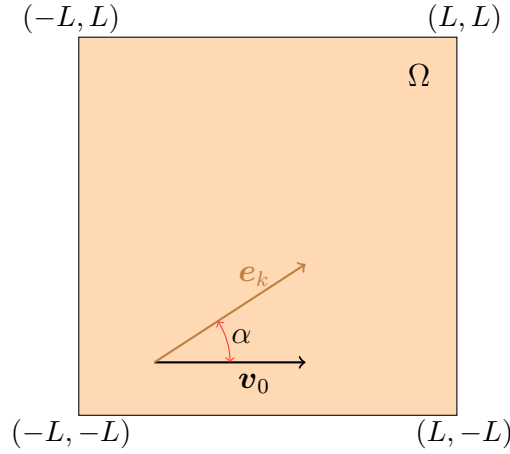


Figure 5.1: Two-dimensional schematic of the uniform-flow benchmark case.

To develop a simple test case for the validation of uniform-flow Galbrun equation, the finite element implementation is tested against simple solutions of the acoustic governing equation in a fluid bounded test.

For that purpose, consider a two-dimensional Cartesian computational domain $\Omega = [-L, L]^2$, with a uniform underlying Eulerian flow, $\mathbf{v}_0$ and an acoustic plane wave of fixed amplitude, i.e., the displacement field $\mathbf{u} = u \exp(i\mathbf{k} \cdot \mathbf{x})\mathbf{e}_k$, propagating with a wavenumber vector $\mathbf{k}$, in an arbitrary direction along $\mathbf{e}_k = \mathbf{k}/|\mathbf{k}|$ as shown in Figure 5.1. If the plane wave is indeed feasible, it should solve the Galbrun equation,

$$\rho_0(-i\omega + \mathbf{v}_0 \cdot \nabla)^2 \mathbf{u} - \nabla(\rho_0 c_0^2 \operatorname{div} \mathbf{u}) = \mathbf{0}$$

or, expanding the operators,

$$\begin{aligned} \rho_0[-\omega^2 - 2i\omega(\mathbf{v}_0 \cdot \nabla) + (\mathbf{v}_0 \cdot \nabla)^2](u \exp(i\mathbf{k} \cdot \mathbf{x})\mathbf{e}_k) \\ - \rho_0 c_0^2 \nabla[\operatorname{div}(u \exp(i\mathbf{k} \cdot \mathbf{x})\mathbf{e}_k)] = \mathbf{0}. \end{aligned}$$

Now, taking into account that the gradient and divergence operators applied on exponential expressions result

$$\begin{aligned}\nabla[u \exp(i\mathbf{k} \cdot \mathbf{x}) \mathbf{e}_k] &= iu|\mathbf{k}| \exp(i\mathbf{k} \cdot \mathbf{x}) (\mathbf{e}_k \otimes \mathbf{e}_k) \\ \text{div}[u \exp(i\mathbf{k} \cdot \mathbf{x}) \mathbf{e}_k] &= iu|\mathbf{k}| \exp(i\mathbf{k} \cdot \mathbf{x}).\end{aligned}$$

Hence, we have

$$\begin{aligned}-\omega^2 u \exp(i\mathbf{k} \cdot \mathbf{x}) \mathbf{e}_k - 2i\omega [iu|\mathbf{k}| \exp(i\mathbf{k} \cdot \mathbf{x}) (\mathbf{e}_k \otimes \mathbf{e}_k)] \mathbf{v}_0 \\ + \nabla [iu|\mathbf{k}| \exp(i\mathbf{k} \cdot \mathbf{x}) (\mathbf{e}_k \otimes \mathbf{e}_k)] \mathbf{v}_0 \\ - c_0^2 \nabla [iu|\mathbf{k}| \exp(i\mathbf{k} \cdot \mathbf{x})] = \mathbf{0},\end{aligned}$$

or equivalently,

$$\begin{aligned}-\omega^2 u \exp(i\mathbf{k} \cdot \mathbf{x}) \mathbf{e}_k + 2\omega u |\mathbf{k}| \exp(i\mathbf{k} \cdot \mathbf{x}) (\mathbf{v}_0 \cdot \mathbf{e}_k) \mathbf{e}_k \\ + \nabla [iu|\mathbf{k}| \exp(i\mathbf{k} \cdot \mathbf{x}) (\mathbf{v}_0 \cdot \mathbf{e}_k) \mathbf{e}_k] \mathbf{v}_0 \\ + c_0^2 u |\mathbf{k}| \exp(i\mathbf{k} \cdot \mathbf{x}) \mathbf{k} = \mathbf{0}.\end{aligned}$$

Dividing by the scalar, $u \exp(i\mathbf{k} \cdot \mathbf{x}) \neq 0$, it holds

$$\left[-\omega^2 + 2\omega |\mathbf{k}| (\mathbf{v}_0 \cdot \mathbf{e}_k) - |\mathbf{k}|^2 (\mathbf{v}_0 \cdot \mathbf{e}_k)^2 + c_0^2 |\mathbf{k}|^2 \right] \mathbf{e}_k = \mathbf{0}.$$

The equation stated above only holds if

$$-\left[\omega - |\mathbf{k}| (\mathbf{v}_0 \cdot \mathbf{e}_k) \right]^2 + c_0^2 |\mathbf{k}|^2 = 0,$$

what implies that $\omega - |\mathbf{k}| (\mathbf{v}_0 \cdot \mathbf{e}_k) = c_0 |\mathbf{k}|$. Since $\mathbf{v}_0 \cdot \mathbf{e}_k = |\mathbf{v}_0| \cos \alpha$, the magnitude of the wavenumber vector can be obtained as

$$|\mathbf{k}| = \frac{\omega}{c_0 + |\mathbf{v}_0| \cos \alpha}. \quad (5.7.1)$$

Hence, the assumed displacement field is a solution of the Galbrun equation if Eq. (5.7.1) is satisfied. Notice that the dispersion relation state above express the relation between the modulus of the wavenumber vector and the angular frequency value, but it holds despite the value of amplitude u .

Remark 5.7.1. *Doppler effect* This dispersion relation is also physically interpreted as the wavenumber associated with a "modulated frequency", $\omega' = |\mathbf{k}|c_0$, such that,

$$\frac{\omega'}{c_0} = \frac{\omega}{c_0 + |\mathbf{v}_0| \cos \alpha} \quad \text{or} \quad \frac{\omega'}{\omega} = \frac{c_0}{c_0 + |\mathbf{v}_0| \cos \alpha}.$$

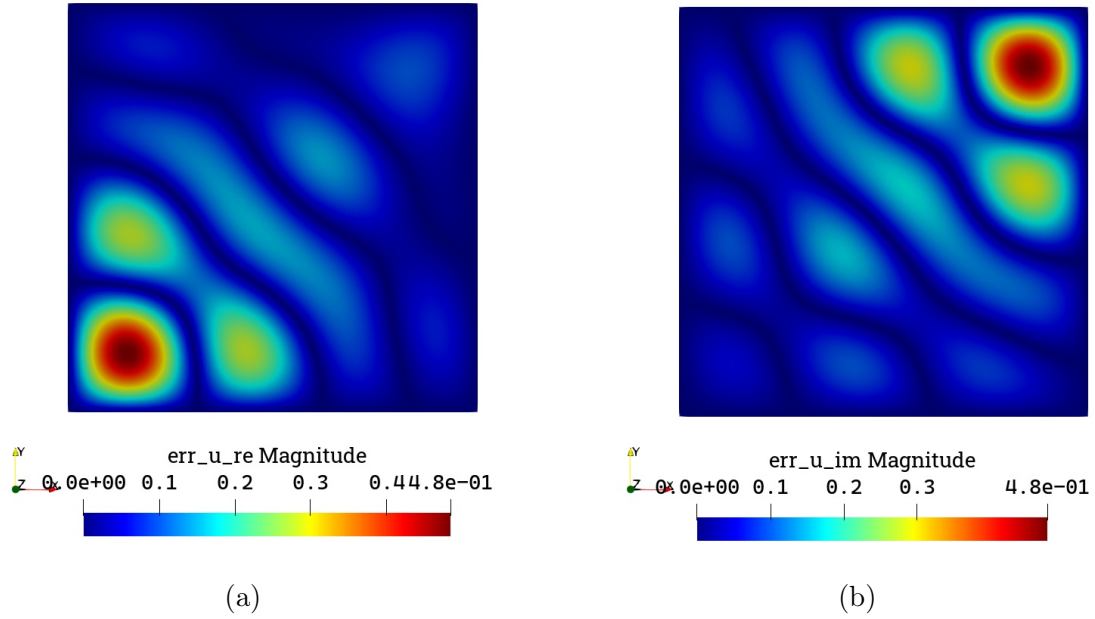


Figure 5.2: The real (left) and scattered part (right) of the error in the displacement field for 800 Hz frequency

The exact solution derived above from the Galbrun equation has been used to construct a benchmark case for validating the finite element implementation with uniform flow in the augmented formulation, where the derived exact solution lead to $\boldsymbol{\psi} = \text{curl } \boldsymbol{u}$. The Dirichlet boundary conditions have been chosen accordingly with the values of the exact solution and the wavenumber given by Eq. (5.7.1).

This benchmark case is implemented for computational domain of $\Omega \in [0, 1] \times [0, 1]$ with a uniform $\boldsymbol{v}_0 = (1.0, 1.0)^t$ for a two dimensional square domain. An incident plane wave field of unit magnitude is considered at a $\pi/4$ rad angle to the horizontal axis. The exact solution is known and the absolute error fields computed are shown in Figure 5.2 as $\xi(\boldsymbol{u}_h) = \boldsymbol{u}_h - \boldsymbol{u}$. The $L^2(\Omega)$ -norm of the errors are also computed for the domain for different grades of mesh refinement and Figure 5.3 highlights its variation with the mesh refinement for few frequencies. The error converges quadratically with mesh refinement in accordance the exact accuracy of the linear Lagrange finite element discretization.

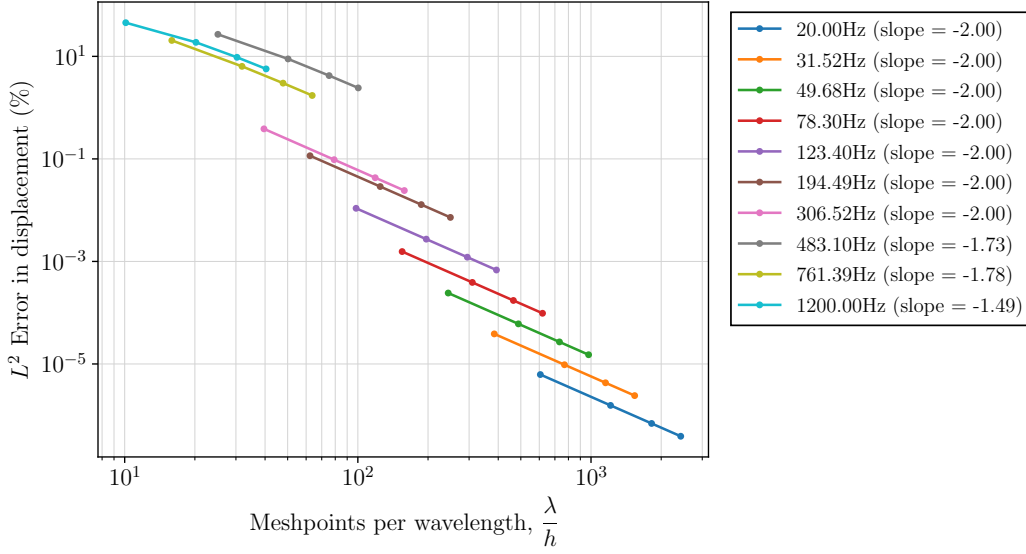


Figure 5.3: $L^2(\Omega)$ errors in displacement field for the two-dimensional augmented Galbrun model.

5.7.2 Uniform-flow Galbrun model with Perfectly Matched Layers

To validate the finite element implementation of the augmented Galbrun model in combination with the Perfectly Matched Layer technique, a simple two-dimensional benchmark has been developed.

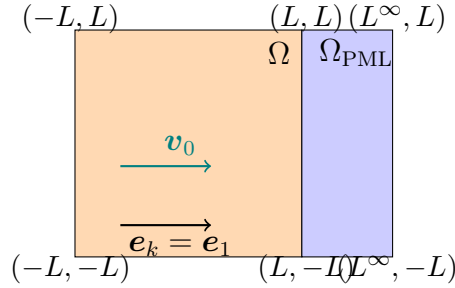


Figure 5.4: Two-dimensional vertical cut of a three-dimensional Cartesian domain with a PML layer only along the x_1 -direction.

With this purpose, consider the a two-dimensional Cartesian computational domain $\Omega = [-L, L]^2$ with a Cartesian PML only along one direction (for instance, $\mathbf{e}_1$), i.e., $\Omega_{\text{PML}} = [L, L^\infty] \times [-L, L]$, with a uniform underlying Eulerian flow, $\mathbf{v}_0 = v_0 \mathbf{e}_1$ and an acoustic plane wave of fixed amplitude, i.e., the displacement field $\mathbf{u} = u \exp(i\mathbf{k} \cdot \mathbf{x}) \mathbf{e}_1$, propagating with a wavenumber vector $\mathbf{k}$, along $\mathbf{e}_1$.

Since the spatial coordinates need to be stretched in the PML, the plane wave within a PML should morph to $\mathbf{u}_{\text{PML}} = u \exp(i\mathbf{k} \cdot \tilde{\mathbf{x}}) \mathbf{e}_1$. Now to verify if this holds, let us consider

the PML governing equation

$$\begin{aligned} & -\rho_0\omega^2\mathbf{u} - 2i\rho_0\omega(\nabla\mathbf{u})\mathbf{H}^{-1}\mathbf{v}_0 \\ & + \rho_0\nabla[(\nabla\mathbf{u})\mathbf{H}^{-1}\mathbf{v}_0]\mathbf{H}^{-1}\mathbf{v}_0 - \rho_0c_0^2\mathbf{H}^{-1}\nabla(\mathbf{H}^{-1}:\nabla\mathbf{u}) = \mathbf{0} \end{aligned}$$

and check for the feasibility of the solution $\mathbf{u}_{\text{PML}} = u \exp(i\mathbf{k} \cdot \tilde{\mathbf{x}})\mathbf{e}_1$. Inserting the previous function in the PML governing equation, each term can be computed straightforwardly as follows:

$$\begin{aligned} \nabla(\mathbf{u}_{\text{PML}}) &= iu|\mathbf{k}|\gamma_1 \exp(i\mathbf{k} \cdot \tilde{\mathbf{x}})(\mathbf{e}_1 \otimes \mathbf{e}_1), \\ \mathbf{H}^{-1}:\mathbf{u}_{\text{PML}} &= iu|\mathbf{k}| \exp(i\mathbf{k} \cdot \tilde{\mathbf{x}}), \\ \nabla\mathbf{u}_{\text{PML}}\mathbf{H}^{-1}\mathbf{v}_0 &= iuv_0|\mathbf{k}| \exp(i\mathbf{k} \cdot \tilde{\mathbf{x}})\mathbf{e}_1, \\ \nabla[\nabla\mathbf{u}_{\text{PML}}\mathbf{H}^{-1}\mathbf{v}_0]\mathbf{H}^{-1}\mathbf{v}_0 &= -uv_0^2|\mathbf{k}|^2 \exp(i\mathbf{k} \cdot \tilde{\mathbf{x}})\mathbf{e}_1, \\ \mathbf{H}^{-1}\nabla(\mathbf{H}^{-1}:\nabla\mathbf{u}_{\text{PML}}) &= -u|\mathbf{k}|^2 \exp(i\mathbf{k} \cdot \tilde{\mathbf{x}})\mathbf{e}_1. \end{aligned}$$

Substituting these expressions in the left-hand side of the PML governing equation, it results

$$\begin{aligned} & -\rho_0\omega^2u \exp(i\mathbf{k} \cdot \tilde{\mathbf{x}})\mathbf{e}_1 - 2i\rho_0\omega[iuv_0|\mathbf{k}| \exp(i\mathbf{k} \cdot \tilde{\mathbf{x}})\mathbf{e}_1] \\ & + \rho_0[-uv_0^2|\mathbf{k}|^2 \exp(i\mathbf{k} \cdot \tilde{\mathbf{x}})\mathbf{e}_1] - \rho_0c_0^2[-u|\mathbf{k}|^2 \exp(i\mathbf{k} \cdot \tilde{\mathbf{x}})\mathbf{e}_1] = \mathbf{0}, \end{aligned}$$

which can be simplified to obtain

$$\begin{aligned} & \rho_0u \exp(i\mathbf{k} \cdot \tilde{\mathbf{x}}) \left[-\omega^2 + 2\omega v_0|\mathbf{k}| - v_0^2|\mathbf{k}|^2 + c_0^2|\mathbf{k}|^2 \right] \mathbf{e}_1 \\ & = \rho_0u \exp(i\mathbf{k} \cdot \tilde{\mathbf{x}}) \left[-(\omega - v_0|\mathbf{k}|)^2 + c_0^2|\mathbf{k}|^2 \right] \mathbf{e}_1 = \mathbf{0}. \end{aligned}$$

Now since, in the convected plane wave satisfies the dispersion relation (derived in the previous benchmark case),

$$|\mathbf{k}| = \frac{\omega}{c_0 + v_0},$$

then the term $-(\omega - v_0|\mathbf{k}|)^2 + c_0^2|\mathbf{k}|^2 = 0$, and hence, the stretched plane wave satisfies the governing equation of the PML layer. Hence, to obtain the plane-wave as a solution of this benchmark problem, the boundary conditions on the top and bottom boundaries of the PML subdomain has to be chosen accordingly with the expression $\mathbf{u} = u \exp(i\mathbf{k} \cdot \tilde{\mathbf{x}})\mathbf{e}_1$, where $\tilde{\mathbf{x}} = (\tilde{x}_1, x_2, x_3)$ being $\tilde{x}_1$ the complex-valued change of variable used to define the PML layer.

Figure 5.6 shows the absolute error fields for each component of the real and imaginary part of the displacement field in this two-dimensional benchmark case. It is observed that the errors are accumulated on a neighbourhood the PML layer.

To check the convergence behaviour of the finite element discretization, a sequence of refined meshes has been used to solve the same two-dimensional benchmark case. The overall trend follows the typical convergence $O(h^2)$, however, it is also observed in Figure 5.6 that if the mesh is enough refined the order of convergence is deteriorated due mainly to the spurious reflections generated by a PML layer of finite thickness at lower frequencies.

Finally, notice that the exact solution derived for the present benchmark is also valid for a three-dimensional setting. Hence, the same convergence analysis has been performed for the three-dimensional version of the same benchmark test. From Figure 5.7, the conclusions and the overall behaviour is analogous to the trend observed in the two-dimensional case.

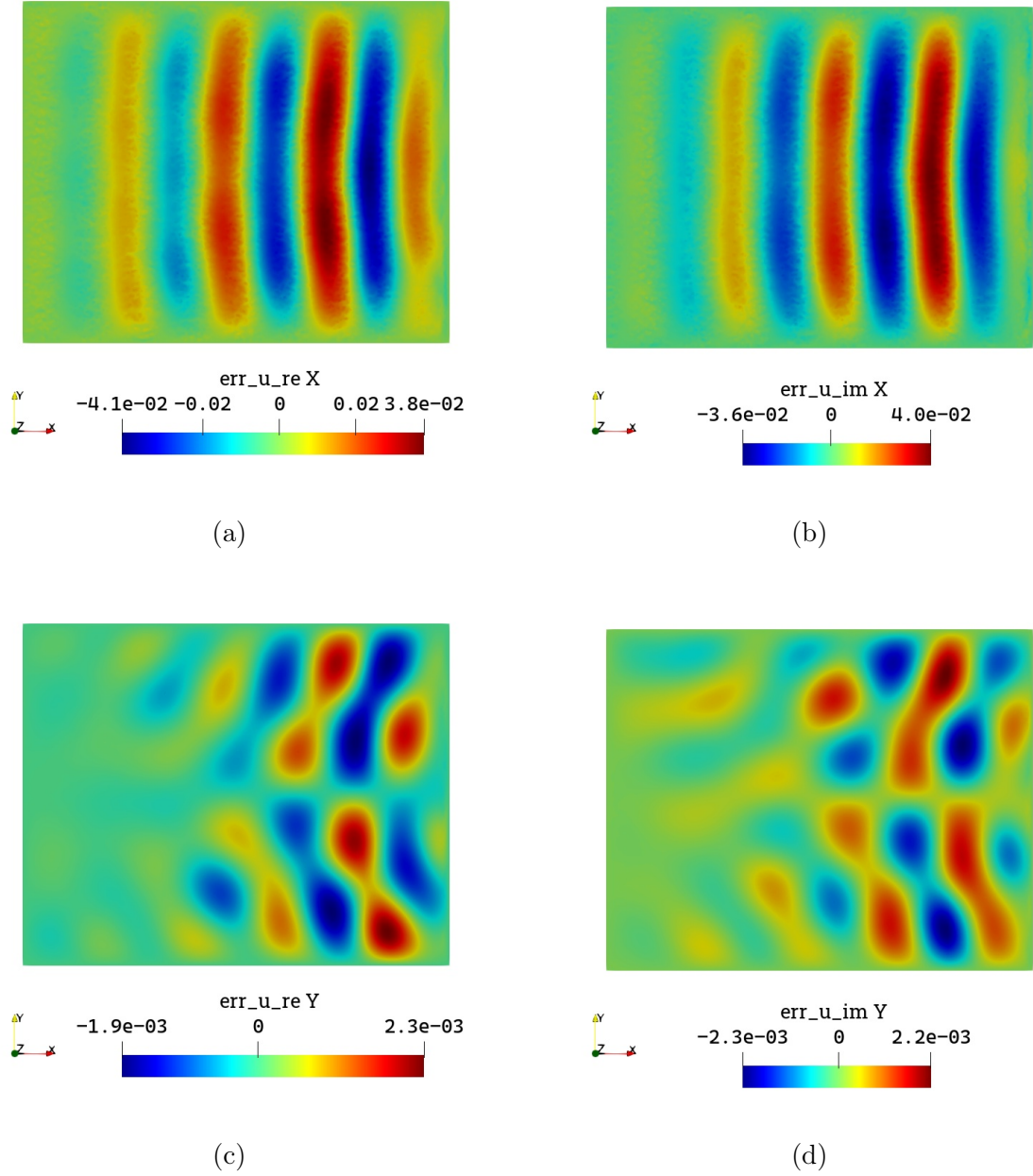


Figure 5.5: The real (left) and scattered part (right) of the error in the displacement field for 1200 Hz frequency.

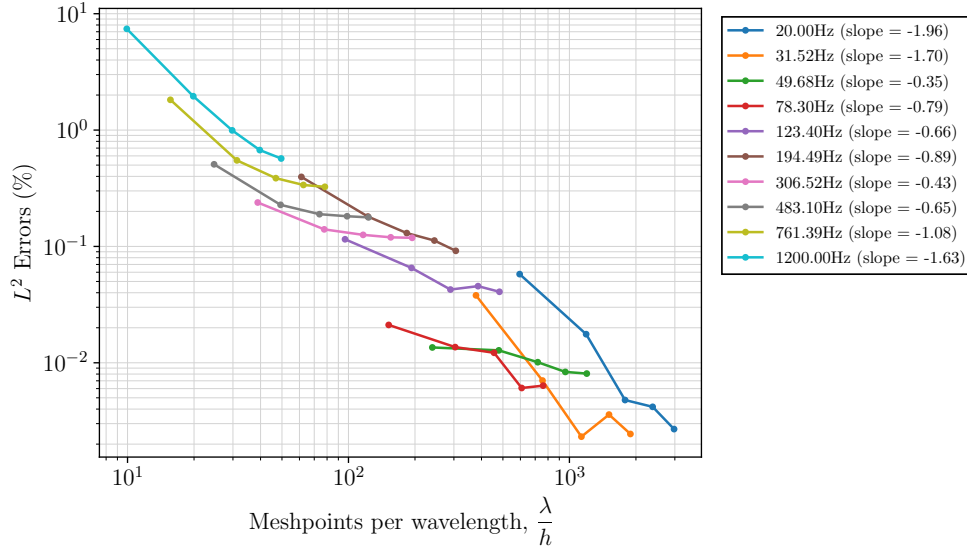


Figure 5.6: $L^2(\Omega)$ -errors in displacement field for the two-dimensional augmented Galbrun model with a Perfectly Matched Layer along the x_1 -direction.

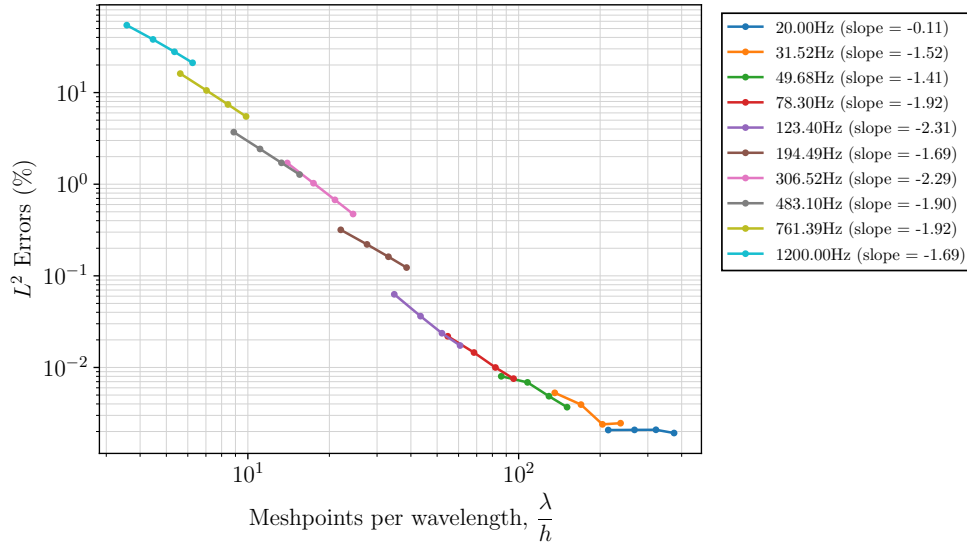


Figure 5.7: $L^2(\Omega)$ -errors in displacement field for the three-dimensional augmented Galbrun model with a Perfectly Matched Layer along the x_1 -direction.

5.7.3 Scattered field under the presence of non-uniform flow

The objective of this section is to develop a scattering problem along the lines of a benchmark case included in [26] and recreate qualitatively the numerical solution reported in that reference. Accordingly, the finite element implementation of the low-Mach number approximation has been used to compute a finite element discrete solution of the classical Galbrun model.

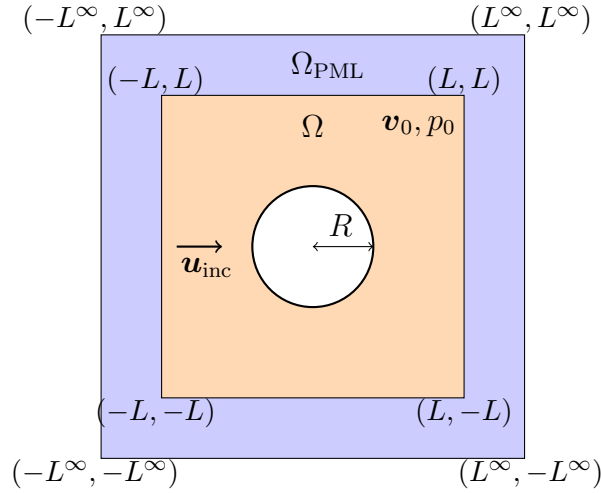


Figure 5.8: Two-dimensional computational domain of the scattering problem surrounded with PML layers

With that purpose, consider a computational domain $\Omega = [L, L]^2$ with an acoustic fluid of density ρ_0 . The fluid is flowing with a laminar flow field $\mathbf{v}_0$ and a pressure field p_0 as shown in Figure 5.8. The laminar flow field is computed as a result of a potential flow around a circle for a inlet underlying flow velocity equal to $v_\infty \mathbf{e}_1$. In this scattering problem, the incident planewave is given by

$$\mathbf{u}^{\text{inc}} = u \exp(i\mathbf{k} \cdot \mathbf{x}) \mathbf{e}_k \quad \text{with } |\mathbf{k}| = \frac{\omega}{c_0 + v_\infty}.$$

In [26], the Eulerian description of the velocity perturbation is computed, so the same physical quantity should be post-processed from the displacement field computed with the Galbrun model. More precisely, the perturbation of the Eulerian velocity can be approximated from the displacement field involved in the Galbrun model by

$$\mathbf{v}_E = -i\omega \mathbf{u} + \nabla \mathbf{u} \cdot \mathbf{v}_0 - \nabla \mathbf{v}_0 \cdot \mathbf{u},$$

and hence it can be computed as a post-processing physical quantity in any numerical simulation of the Galbrun model family.

This benchmark scenario has been implemented for a two-dimensional computational domain with $L = 3$, $R = 0.5$, $L^\infty = 1$ and the underlying flow has been computed using

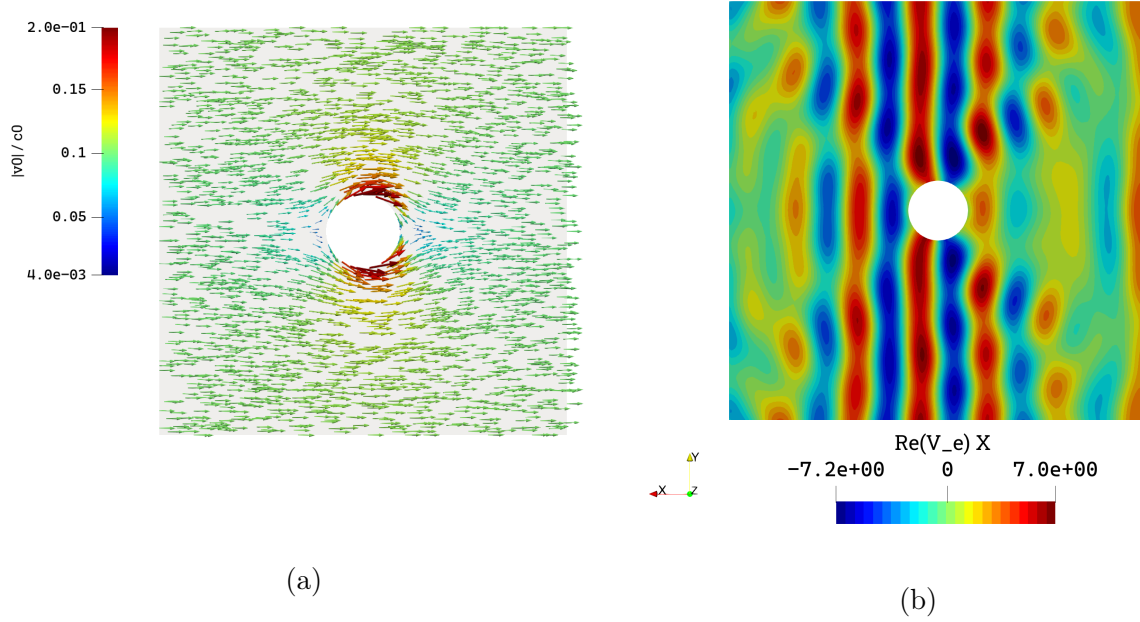


Figure 5.9: Potential flow around the obstacle (left) and the x_1 -component of the Eulerian velocity field (right)

a potential approximation with an inlet velocity $\mathbf{v}_0 = 0.1c_F$. The use of the potential approximation to compute the flow around the obstacle has been also reported in the original article [26]. Figure 5.9(a) shows the result of the potential flow along the x_1 direction.

A plane wave of amplitude $u = 1$ is considered incident on the circular obstacle. The low-Mach number approximation of the Galbrun model has been discretized by the finite element method and solved for a frequency of 309 Hz. The x_1 -component of the resulting Eulerian velocity field is displayed in Figure 5.9(b). A direct inspection comparing these results with the numerical outputs reported in Figure 1(center) in [26] shows a qualitative agreement between both numerical simulations.

5.8 Results and Discussion

The finite element implementation of the low-Mach number approximation of the coupled problem involving the fluid Galbrun model and the Darcy-Forchheimer-Galbrun model for porous materials with rigid solid frame has been used to compute the scattered and total field generated by an incident planewave (of unit amplitude) travelling in the x_1 -direction. To illustrate the effects of the fluid flow and the insertion of a porous material surrounding the structure of the PU probe a wide variety of physical settings have been simulated and reported in the sections below.

5.8.1 Results with the low-Mach number Galbrun Model

Firstly, the finite element implementation of the low-Mach number approximation has been used to solve numerically the fluid Galbrun model. All the simulations have been computed with a frequency of 2000 Hz.

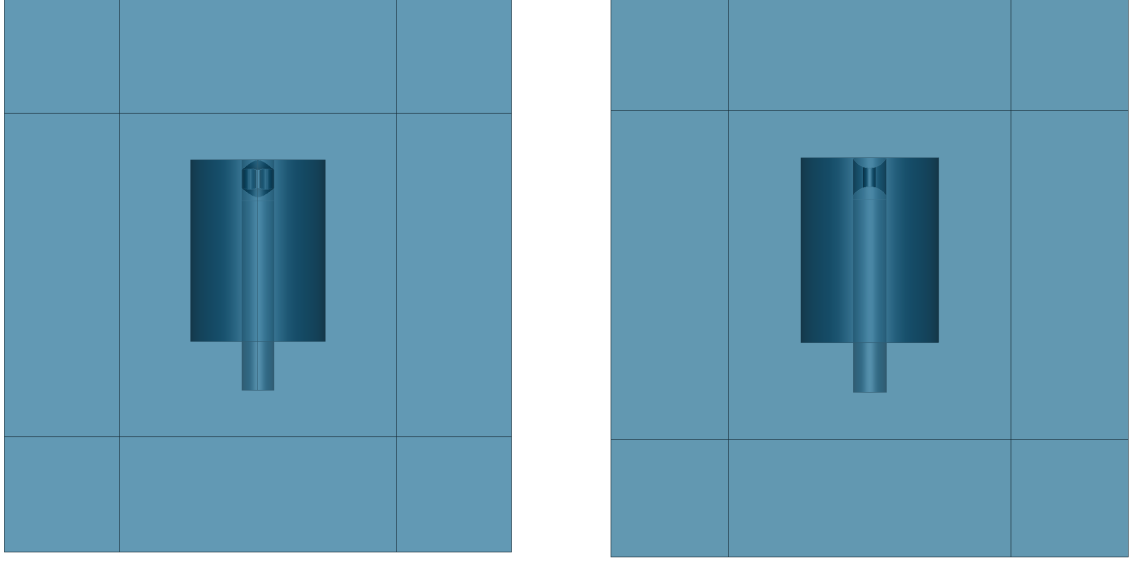


Figure 5.10: Domain geometry along the x_1 (left) and x_2 (right) directions.

Two different incompressible underlying flows have been considered in the numerical simulations, which are computed with a given inlet velocities equal to 0.02 m/s and 0.05 m/s. In both cases, the magnitude of the real and the imaginary parts of the scattered and total displacement fields are reported. Notice also that the scattered fields are plotted not only on the physical domain but also in the PML region, where it can be observed an exponential decay of the magnitude of the displacement field. In the case of the plots reporting the total displacement field, the values of the magnitude of the real and the imaginary part are displayed on the physical domain of the problem.

More precisely, Figure 5.13 and 5.13 show the scattered and the total displacement fields for an inlet flow velocity of 0.02 m/s.

Analogously, Figure 5.15 and 5.15 show the scattered and the total displacement fields for an inlet flow velocity of 0.05 m/s. If the numerical results are compared with respect to

	Characteristic Length, D(m)	Kinematic Viscosity, $\nu(\text{m}^2\text{s}^{-1})$	Velocity, U(m/s)	Reynold's Number, Re
Probe diameter	1.26E-02	1.52E-05	0.01	8.311
			0.02	16.623
			0.05	41.557

Figure 5.11: Reynolds Number of the simulations

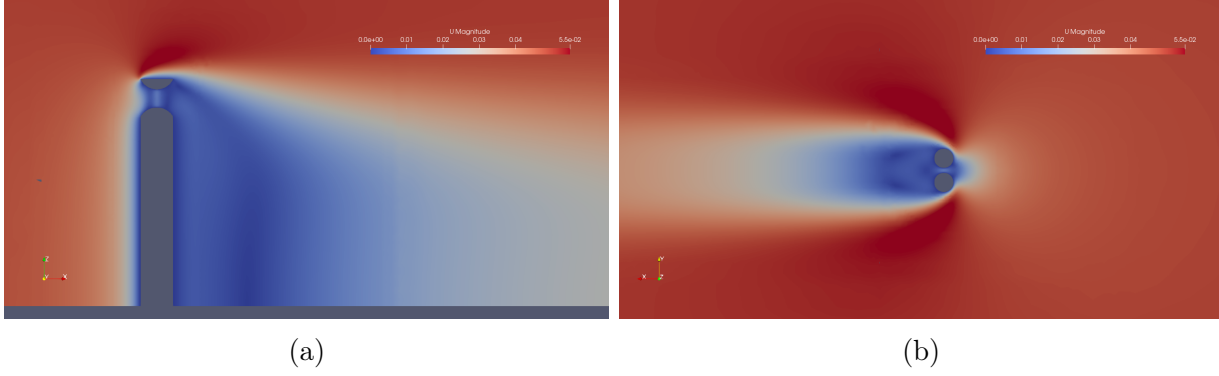


Figure 5.12: Slices of the steady laminar flow results for 0.05m/s inlet velocity in the y-plane (a) and on the z-plane between the pillars (b)

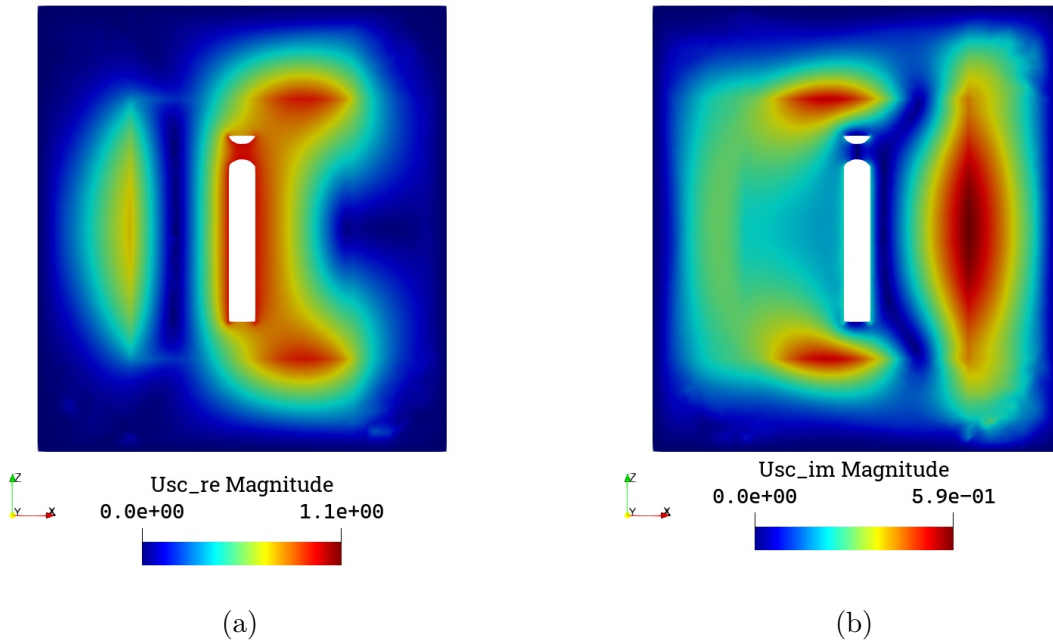


Figure 5.13: The scattered displacement fields for a frequency of 2000Hz for an incident laminar flow of $0.02ms^{-1}$ around the PU probe.

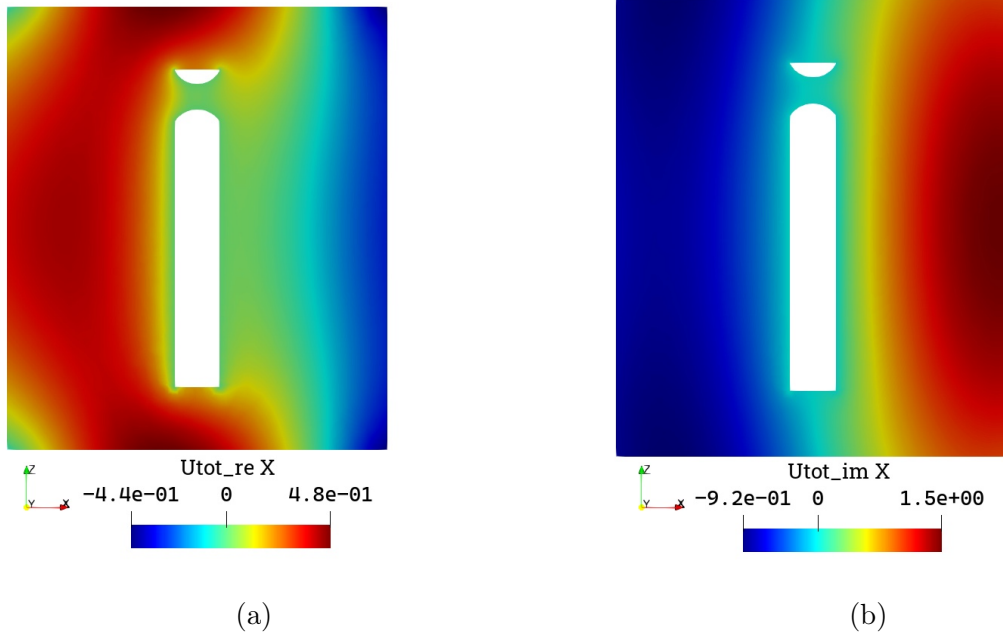


Figure 5.14: The total displacement fields for a frequency of 2000Hz for an incident laminar flow of 0.02ms^{-1} around the PU probe.

these two different underlying velocities, it can be concluded that there are no qualitative differences between both simulations at this regime (since the Reynolds and the Mach number is small).

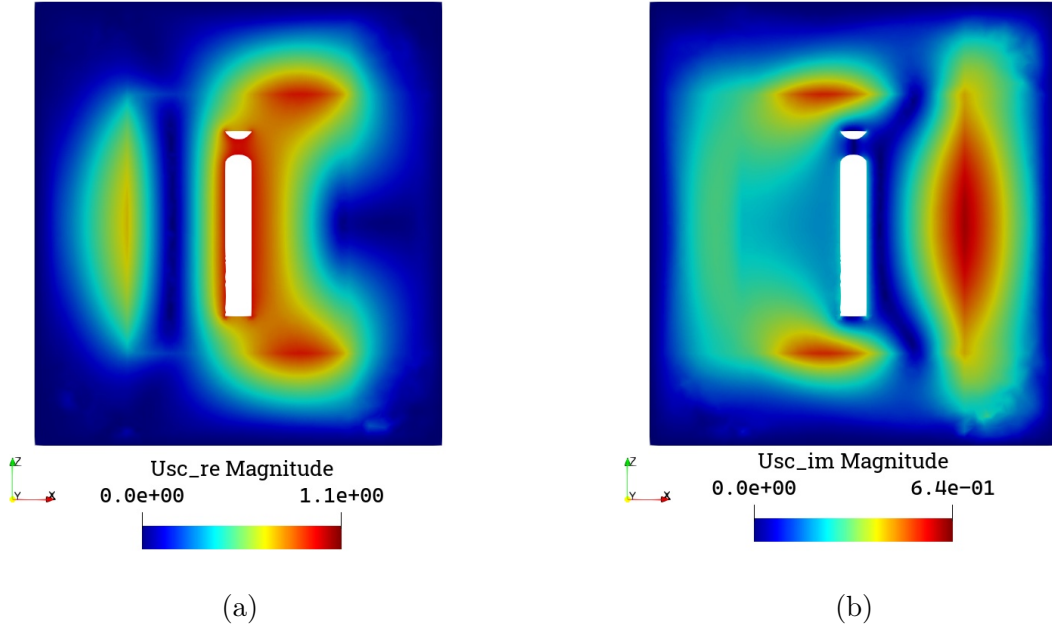


Figure 5.15: The scattered displacement fields for a frequency of 2000Hz for an incident laminar flow of $0.05ms^{-1}$ around the PU probe

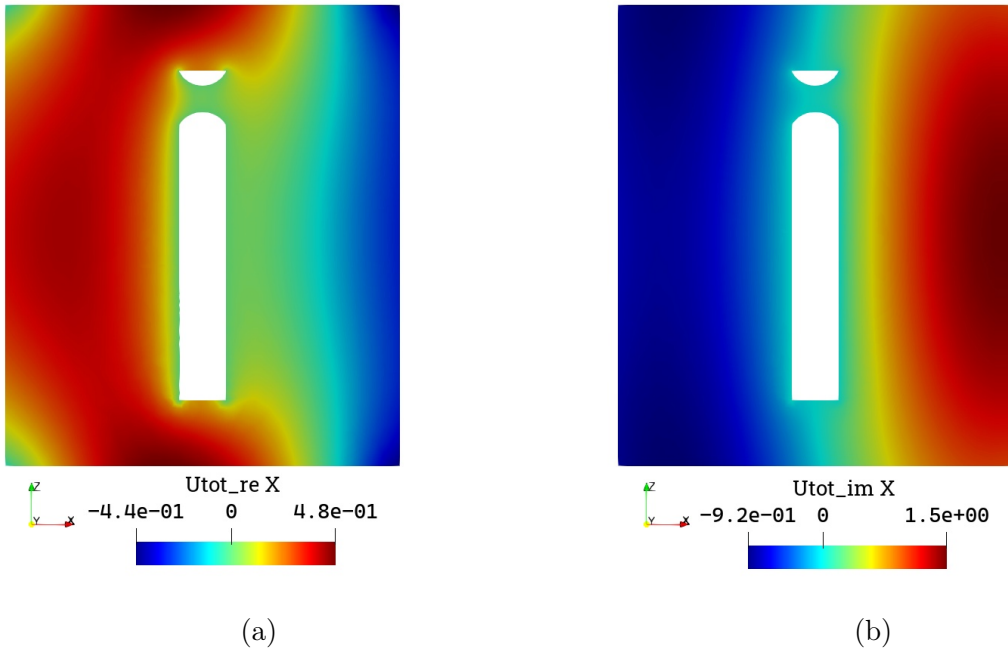


Figure 5.16: The total displacement fields for a frequency of 2000Hz for an incident laminar flow of $0.05ms^{-1}$ around the PU probe.

5.8.2 Results with the low-Mach number Darcy-Forcheimer-Galbrun Model

Second, the finite element implementation of the low-Mach number approximation has been used to solve numerically the coupled scattering problem involving the fluid Galbrun model and the Darcy-Forcheimer-Galbrun model. All the simulations have been computed with a frequency of 2000 Hz and neglecting the Forcheimer correction.

Two different incompressible underlying flows have been considered in the numerical simulations, which are computed with a given inlet velocities equal to 0.01 m/s and 0.05 m/s. In both cases, the magnitude of the real and the imaginary parts of the scattered and total displacement fields are reported. Notice again that the scattered fields are plotted not only on the physical domain but also in the PML region, where it can be observed an exponential decay of the magnitude of the displacement field. In the case of the plots reporting the total displacement field, the values of the magnitude of the real and the imaginary part are displayed on the physical domain of the problem.

To evaluate the impact of different porous materials acting as windscreens of the PU probe, two different porous materials have been considered with flow resistivity values $\sigma = 5000 \text{ Nsm}^{-4}$ and 15000 Nsm^{-4} . Let us recall the relation between the flow resistivity value and the α coefficient in the Darcy-Forcheimer-Galbrun model is given by $\sigma = \phi \rho_F \alpha$, being ρ_F the mass density and ϕ the porosity.

More precisely, Figures 5.17 and 5.18 show the scattered and the total displacement fields for an inlet flow velocity of 0.01 m/s and $\sigma = 5000 \text{ Nsm}^{-4}$. If we compare those numerical results with those numerical simulations reported in Figures 5.21 and 5.22, it can be observed that the differences are negligible, shown again the small effect of the underlying velocity at this low velocity regime. A similar conclusion can be obtained by performing a comparison between Figures 5.19 and 5.20 and Figures 5.23 and 5.24, which have in common the same flow resistivity value $\sigma = 15000 \text{ Nsm}^{-4}$ but different inlet flow velocity, 0.01 m/s and 0.05 m/s.

Finally, let us comment that the effects of inserting the porous windscreen is obvious for any magnitude of the underlying flow. For instance, if we compare the numerical results reported in Figures 5.17 and 5.18 and they are compared with respect to the plots shown in Figures 5.19 and 5.20, a different pattern on the scattered and total displacement field is found. In addition, in the case of higher resistivity flow simulations the diffracted field reaches lower magnitude values on the top region around the cap of the PU probe.

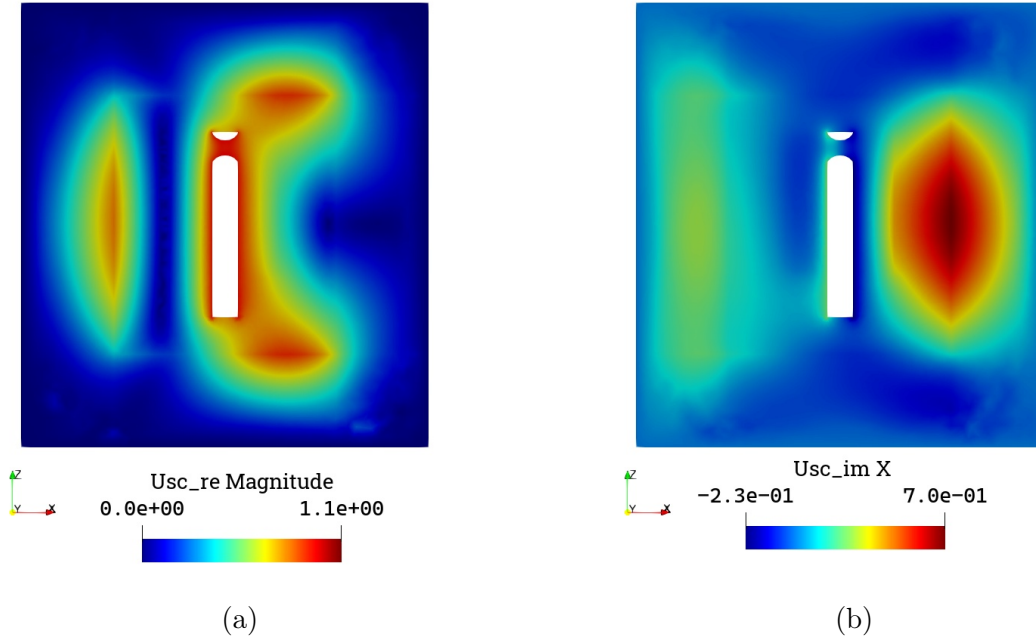


Figure 5.17: The scattered displacement fields for a frequency of 2000Hz for an incident laminar flow of $0.01ms^{-1}$ and a porous flow resistivity of 5000 for the PU probe with a porous windscreen.

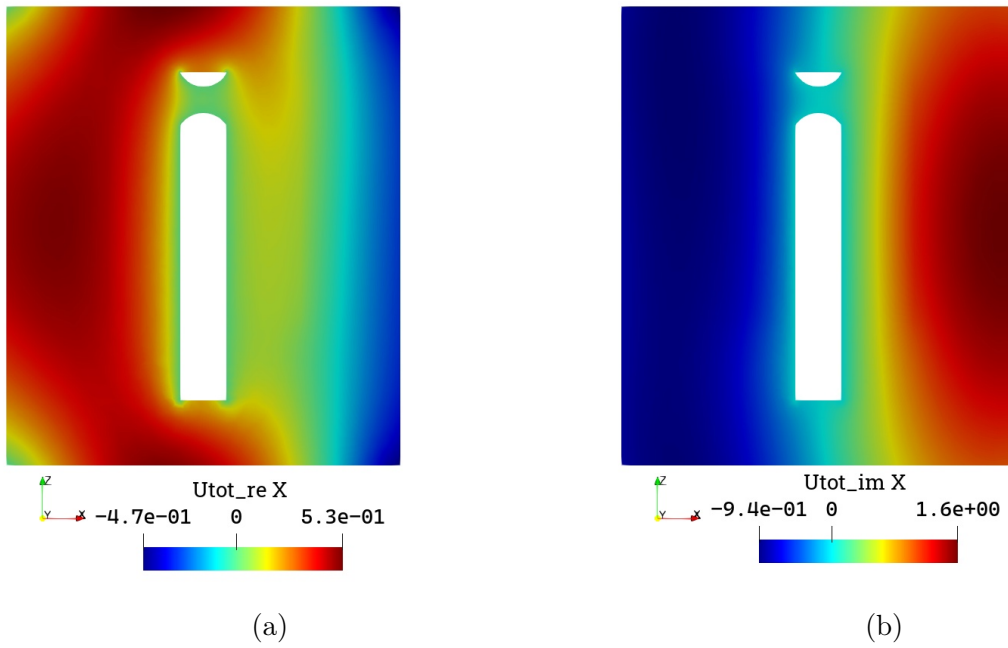


Figure 5.18: The total displacement fields for a frequency of 2000Hz for an incident laminar flow of $0.01ms^{-1}$ and a porous flow resistivity of 5000 for the PU probe with a porous windscreen.

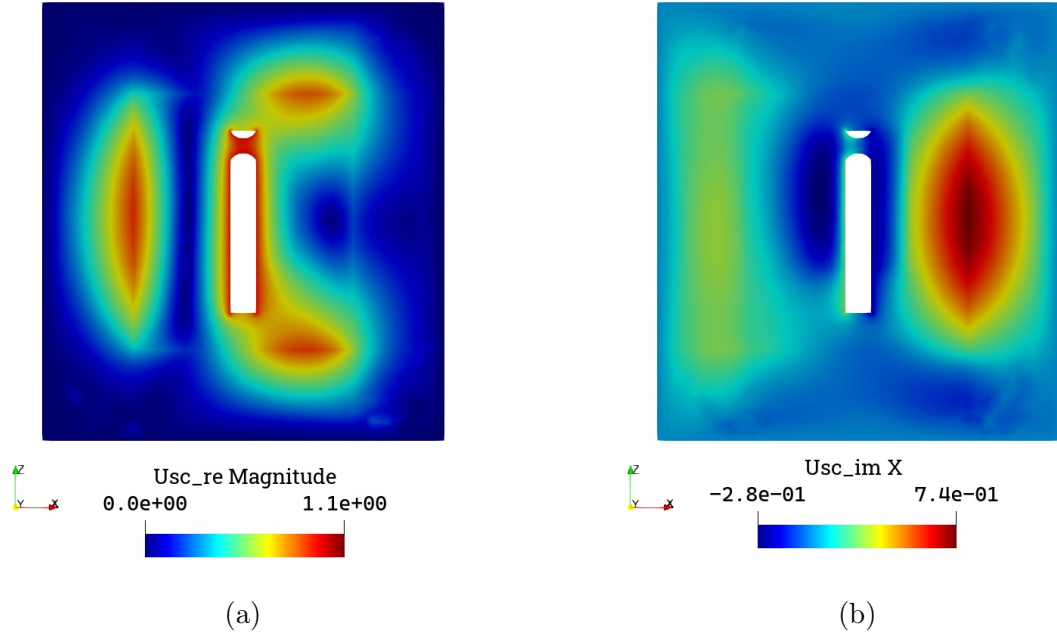


Figure 5.19: The scattered displacement fields for a frequency of 2000Hz for an incident laminar flow of $0.01ms^{-1}$ and a porous flow resistivity of 15000 for the PU probe with a porous windscreen.

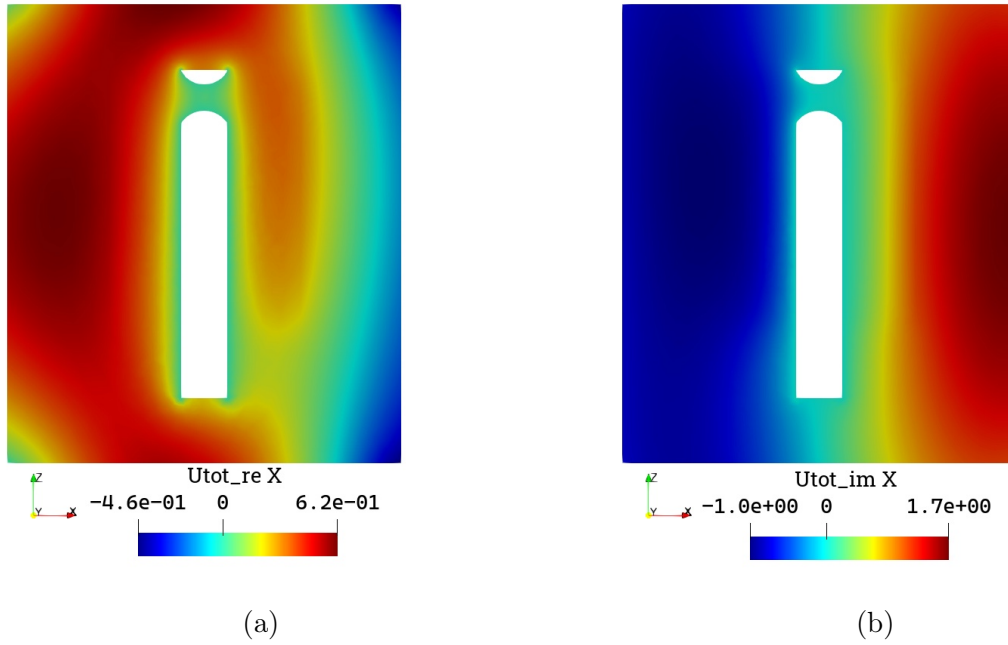


Figure 5.20: The total displacement fields for a frequency of 2000Hz for an incident laminar flow of $0.01ms^{-1}$ and a porous flow resistivity of 15000 for the PU probe with a porous windscreen.

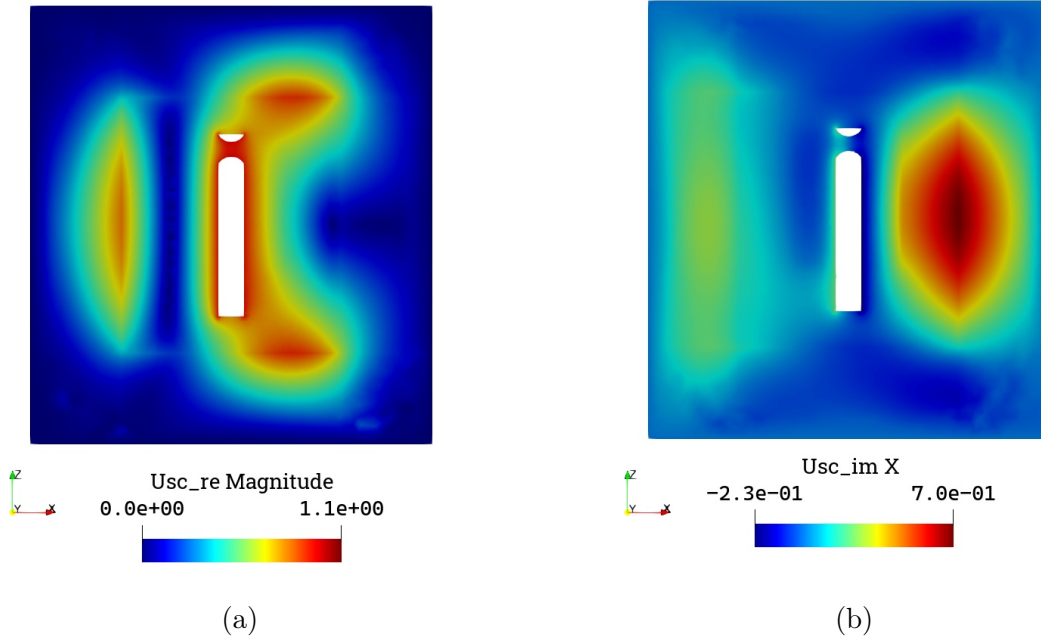


Figure 5.21: The scattered displacement fields for a frequency of 2000Hz for an incident laminar flow of $0.05ms^{-1}$ and a porous flow resistivity of 5000 for the PU probe with a porous windscreen.

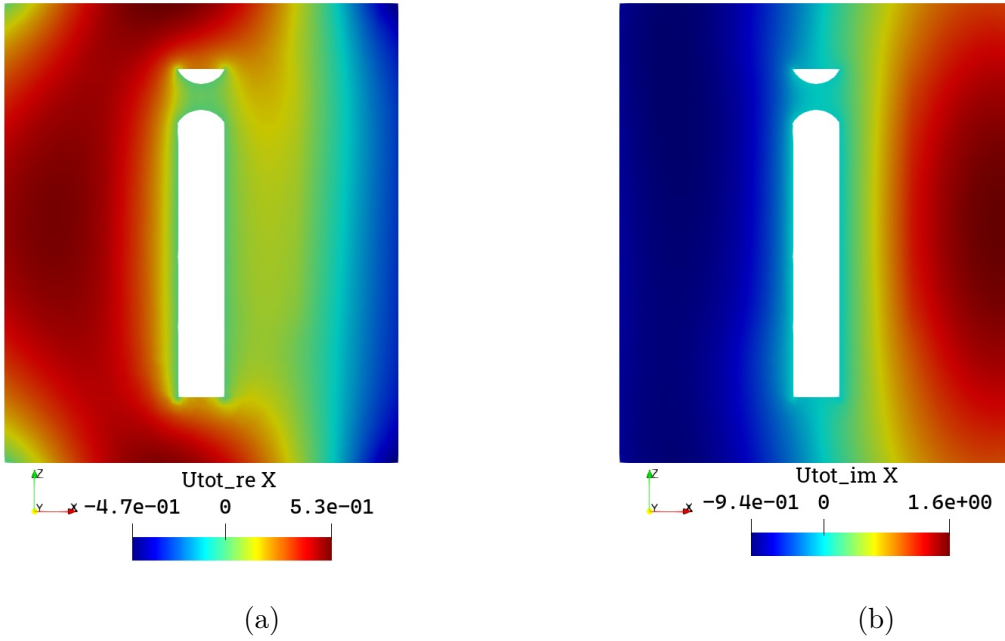


Figure 5.22: The total displacement fields for a frequency of 2000Hz for an incident laminar flow of $0.05ms^{-1}$ and a porous flow resistivity of 5000 for the PU probe with a porous windscreen.

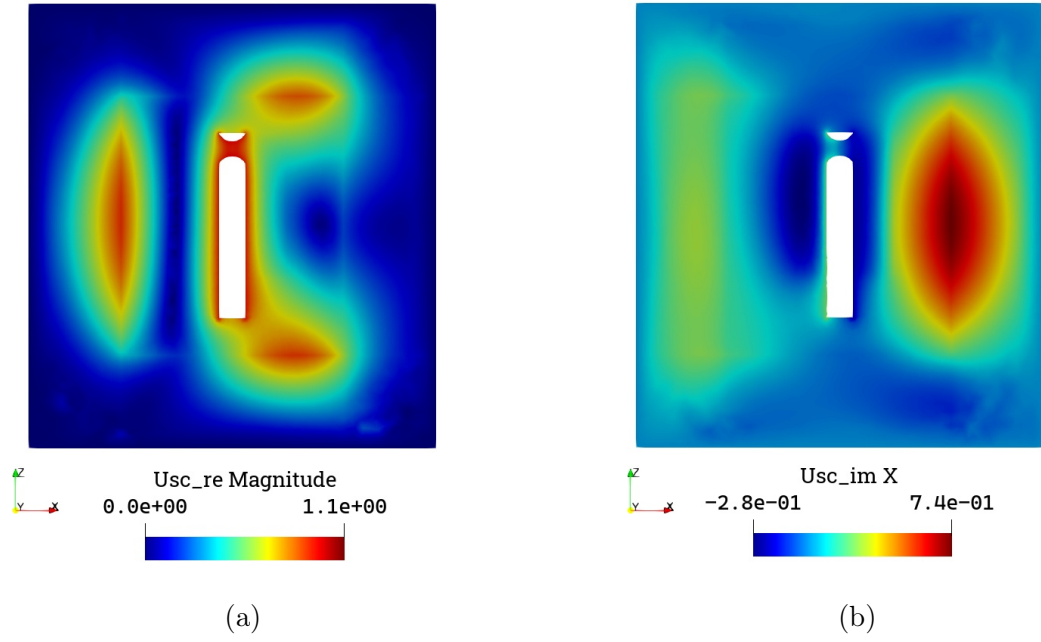


Figure 5.23: The scattered displacement fields for a frequency of 2000Hz for an incident laminar flow of $0.05ms^{-1}$ and a porous flow resistivity of 15000 for the PU probe with a porous windscreen.

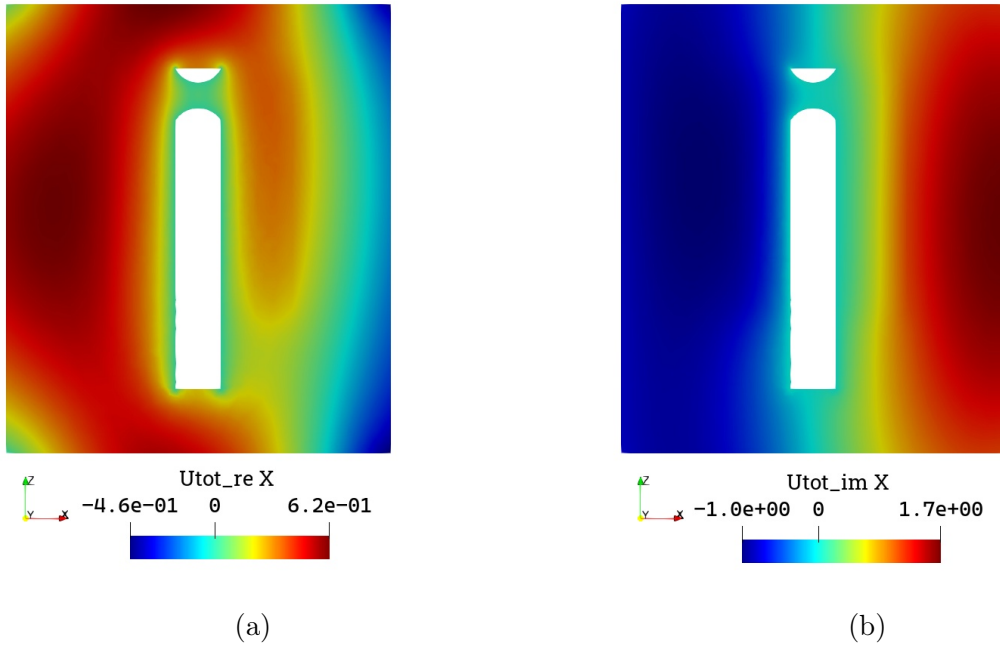


Figure 5.24: The total displacement fields for a frequency of 2000Hz for an incident laminar flow of $0.05ms^{-1}$ and a porous flow resistivity of 15000 for the PU probe with a porous windscreen.

.1 Computation of the underlying flow with the Finite Volume Method

The partial differential equations involved in the Navier-Stokes equations needs discretization to arrive at a linear system of equations to be able to practically solve for a solution. While several techniques offer suitable ways of proceeding numerically, the Finite Volume Method (FVM) is chosen for ease of implementation.

The FVM method discretizes the space and time using an integral form of the governing equations while preserving the accuracy of different conservation laws over each control volume. The basic quantities, such as mass and momentum, is therefore conserved at the discrete level. Any general polyhedral shape is accepted for the control volume with a variable number of neighbours, thus creating an arbitrarily unstructured polygonal mesh. All dependent variables share the same control volume, which is usually called the collocated or non-staggered variable arrangement. The implementation of a finite volume method is beyond the scope of the present work and hence the open-source implementation of `OpenFOAM` is utilized.

Solving the Navier Stokes equation presents two challenges that require special attention: non-linear momentum equations and the pressure-velocity coupling within them. Various strategies exist to solve them and can be broadly classified into a simultaneous or a segregated approach. Solving the simultaneous system bears a huge computational cost and hence the segregated approach is the most followed technique in the literature. The segregated approach solves the multi-variable equations one at a time, with the inter-equation coupling treated in the explicit manner. Non-linear differential equations are linearised before the discretization and the non-linear terms are lagged. [50]

Typically, the temporal discretizations (for those problems where the solution is time-dependent) is performed using the the Crank-Nicholson method. It is unconditionally stable (it is not required to fulfill a restrictive CFL condition in terms of the Courant number) and second order accurate in time. However, if a steady-state problem is being solved iteratively, it is not necessary to fully resolve the linear pressure-velocity coupling, as the changes between consecutive solutions are no longer small. Non-linearity of the system becomes more important, since the effective time-step is much larger. In the used `OpenFOAM` implementation, the SIMPLE algorithm introduced by Patankar is formulated to take advantage of the following facts:

- i) An approximation of the velocity field is obtained by solving the momentum equation. The pressure gradient term is calculated using the pressure distribution from the previous iteration or an initial guess. The equation is under-relaxed in an implicit manner, with the velocity under-relaxation factor α .
- ii) The pressure equation is formulated and solved in order to obtain the new pressure distribution.
- iii) A new set of conservative fluxes is calculated. The coefficients in the velocity operator

are computed with the new set of conservative fluxes. The pressure solution is then under-relaxed in order to take into account the velocity part of the error.

In particular, for steady-state solution the SIMPLE algorithm involves the following steps:

1. Set all field values to some initial guess.
2. Assemble and solve the under-relaxed momentum predictor equation
3. Solve the pressure equation and calculate the conservative fluxes. Update the pressure field with an appropriate under-relaxation. Perform the explicit velocity correction.
4. Solve the other equations in the system using the available fluxes, pressure and velocity fields. In order to improve convergence, under-relax the other equations in an implicit manner.
5. Check the convergence criterion for all equations. If the system is not converged, start a new iteration on step 2.

Each iteration in the SIMPLE algorithm requires the solution of a large linear system (due to the large number of cells used in the discretization by the finite element volume method). Specialized linear solvers are used for that purpose. More precisely, pressure corrections utilized the GAMG (Geometric Agglomerated Algebraic Multigrid) solver with Gauss-Seidel smoother. Velocity computations have been made with the Preconditioned bi-conjugate gradient (PBiCG) solver with the simplified Diagonal-based Incomplete LU (DILU) smoother.

Appendices

Appendix A

Software Implementation of coupled methods in acoustics

A.1 Software implementation

The implementation follows the requirements of the model and may be divided into three main stages viz., mesh generation, solving equations and visualizing solutions. The different stages of the implementation and the overall workflow is illustrated in Figure A.1. The mesh generation stage requires the user inputs on geometrical configuration of the setup. This includes the exact dimensions of the structure, porous layer, fluid domain and PML. Considering that the variational form includes integrals which differ in sub-domains, it is necessary to mark the mesh cells according to region requiring conformality of the mesh with the geometry of sub-domains. Furthermore, user inputs may be needed to suggest local refining of the mesh in a particular region or surface to capture the geometry accurately. The generated mesh also needs to be adapted to the file format compatible with the solver. The solver imports the mesh data and categorizes cells according to the sub-domain regions. It is responsible for implementing the finite-element method - defining the discrete functional space with chosen basis functions and assembling the system of equations before solving them. The solution obtained may also include processing for analysis before being saved in a memory-efficient storage format. Finally, the visualization tool reads the simulated solution from the disk to provide graphical representations aiding the user in deriving information and performing analysis. The following sections explain the usage of each of the stages and the related tools in detail.

A.1.1 Geometry and Meshing

The digital representation of the setup is first done by modeling the geometry and then discretizing it to form a mesh. While several tools and techniques are available for this, the open-source modules offered by SALOME are used in this article, which provides capabilities for interfacing with various numerical simulation tools. It has a flexible cross-

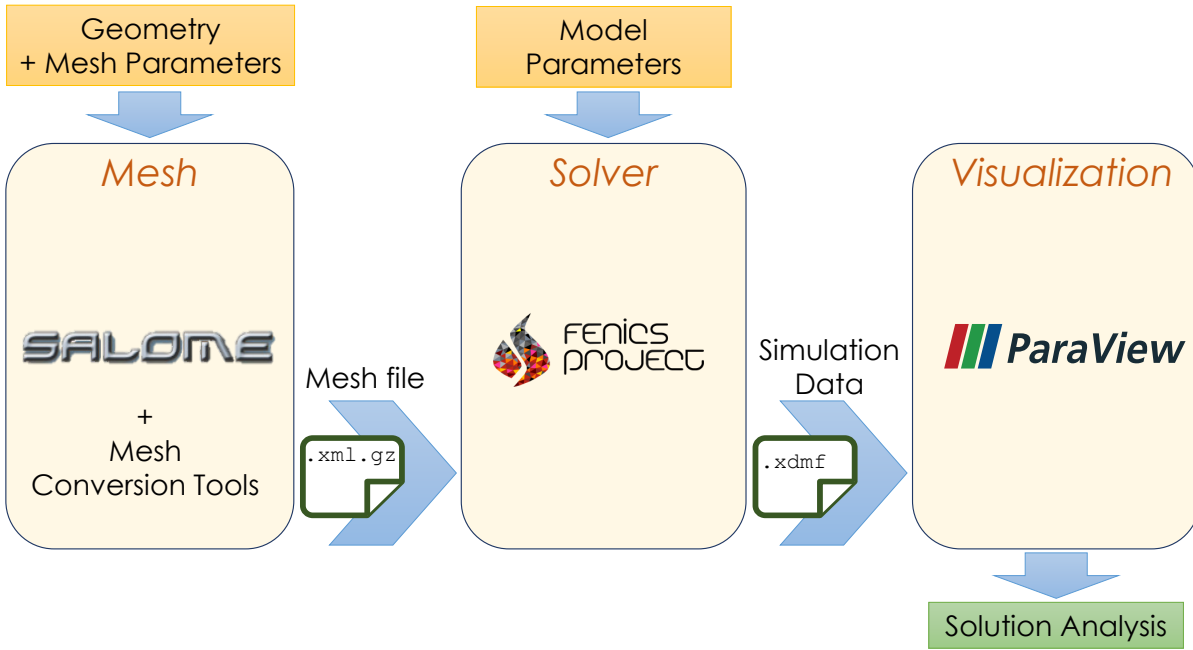


Figure A.1: Workflow representing implementation stages and their interfaces.

platform architecture made of reusable components allowing for customized integration and handling of complex geometrical objects. It allows for creation of geometry and meshes using either (or both) the graphical user interface (GUI) and a text user interface (TUI). The following sections explain the usage of creating geometries and meshes using the TUI, a powerful Python-based scripting interface. The same may also be achieved either in part or entirely by using the GUI. The GUI also allows for exporting the equivalent state in a TUI script.

The TUI provides `geomBuilder` class for the creating and editing geometry. The class instance contains a list of function attributes for operations useful in creating complex geometrical objects. It allows for creating basic objects and primitives in 1D, 2D, 3D; perform boolean operations like fuse, common, cut and section operations; execute extrusion, rotation and other linear operations; create higher order topological objects like solids and compounds grouped from primitives; and implement an advanced partition or gluing between geometrical structures, among others. Table A.1 lists some useful TUI commands available as function attributes, whereas a detailed description along with other functions are available in the documentation [4].

Along with these capabilities, is extremely useful in defining parameters and in making it an. The meshing section is again accessed through another Python module, `smeshBuilder`. It presents different algorithms to create meshes on the basis of geometrical models created by `geomBuilder`. The module also provides control on mesh generation like maintaining conformity between subgroups, splitting, editing, boolean operations and marking. These are

MakeVertex	Creates a vertex
MakeVectorDXDYDZ	Creates a vector
MakeBoxTwoPnt	Creates a Cuboid defined by ends of a body diagonal
MakeSphereR	Creates a Sphere at origin given the radius
MakeCylinder	Creates a Cylinder
TranslateDXDYDZ	Linear translation of a geometrical object
MakeCutList	Boolean operation between two geometrical object
MakePartition	Group various sub-shapes into one geometrical object
SubShapeSortedCentres	Obtain sub-structures of a complex object

Table A.1: Useful Functions in SALOME's `geomBuilder` class

handy in segregating subdomains accurately in a mesh. The NETGEN-1D2D3D algorithm is utilized for the purpose of the project which provides a range of control parameters like tetrahedral or hexahedral mesh elements, specifying the global maximum or minimum size of the edges and also control locally-permissible edge sizes on a lower-order geometrical construct - useful in refining the mesh near a point or a face.

The instance of the `smeshBuilder` class is provided with the geometrical object to mesh and the algorithm specifications. The snippet code below illustrates access to the parameters of NETGEN-1D2D3D algorithm and specify the relevant conditions for the mesh. The most useful functions are listed in table A.1. The mesh is computed after all the parameters are set.

```
from salome.smesh import smeshBuilder
smesh = smeshBuilder.New() #Instantiate the smeshBuilder class
domain_mesh = smesh.Mesh(Domain) #Domain is a geometrical object

# Select type of elements and algorithm to compute them
NETGEN_3D = domain_mesh.Tetrahedron(algo=smeshBuilder.NETGEN_1D2D3D)
NETGEN_3D_Params = NETGEN_3D.Parameters() #Access parameters.
```

The SALOME classes are provided only in its own environment and the scripts are executed through a wrapper offered as,

```
$ salome -t script.py # Runs script.py in SALOME's shell
```

The scripting interface of SALOME is utilized to prepare the geometry and mesh for the problem. The parameters of the sphere is set initially in the script. Once the geometry building module is instantiated, it is conventional to create an origin and the axes. The volumetric spherical object may be created using the `MakeSphereR` function. The porous layer is then created by an intersection of two concentric spheres using the `MakeCutList` function to perform a boolean deletion of the volume of one by another. The two-part

fluid region is also made using a similar boolean operation on a Cartesian box where the "unbounded" domain is truncated.

```

from salome.geom import geomBuilder

# Parameters
L_0 = L_1 = L_2 = 0.2
R_0 = 0.5
R_1 = 2 * R_0
R_2 = 2.25 * R_0

# Create an instance of the geometry builder class
geompy = geomBuilder.New()

# Origin and Axes
O = geompy.MakeVertex(0, 0, 0)
OX = geompy.MakeVectorDXDYDZ(1, 0, 0)
OY = geompy.MakeVectorDXDYDZ(0, 1, 0)
OZ = geompy.MakeVectorDXDYDZ(0, 0, 1)

# Sphere
Sphere = geompy.MakeSphereR(R_0)

# Porous Layer
porous_in = geompy.MakeSphereR(R_1)
porous_out = geompy.MakeSphereR(R_2)
porous_layer = geompy.MakeCutList(porous_out, [porous_in], True)

# Fluid Domain
b0 = geompy.MakeVertex(L_0, L_1, L_2)
b1 = geompy.MakeVertex(-L_0, -L_1, -L_2)
B = geompy.MakeBoxTwoPnt(b0, b1)
Fluid1 = geompy.MakeCutList(B, [porous_out], True)
Fluid2 = geompy.MakeCutList(porous_in, [Sphere], True)

```

While the basic domains of interest are created in an obvious manner outlined above, care is taken in constructing the PML layer. The integration of the variational equation within the PML domain contains the terms H and γ 's which are dependent on the axial orientation. To ensure mesh conformality, the layer is formed by blocked subdomains along each axis. Highlighted below is creation of one such layer enveloping the fluid domain on the x_1 direction,

```

# x0-direction subdomain
be0 = b0
be1 = geompy.MakeVertex(L_0+L_pml, -L_1, -L_2)
Be = geompy.MakeBoxTwoPnt(be0, be1)

```

Similar such blocks are created enveloping the fluid domain along all the directions - two each along the x_1 , x_2 and x_3 directions, four each along x_1x_2 , x_2x_3 and x_3x_1 directions and eight along the corners i.e. the $x_1x_2x_3$ direction. All the layers formed are unified in a single geometrical object achieved by the function attribute `MakePartition` as,

```
Domain = geompy.MakePartition(Domainlist)
```

where, `Domainlist` is a Python list of all the separate subdomains. The function `SubShapeSortedCent` is then used to obtain any sub-structure identifiers, useful in specifying local mesh refinements. Figure A.2a shows a cross-section of the generated geometry.

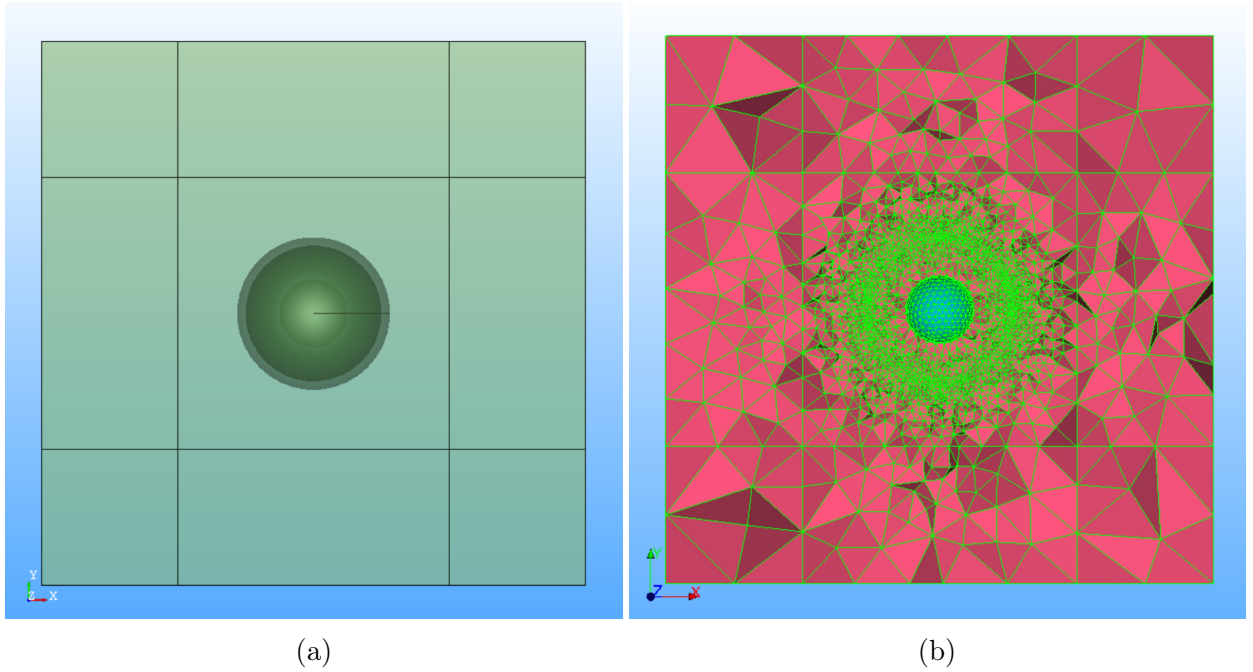


Figure A.2: View of the geometry(left) and the cross-section of the Mesh(right).

The meshing operation is then carried out using the `smesh` module of SALOME as was similarly suggested in A.1.1 . Figure A.2b shows a cross-sectional view of the computed mesh volume along with local refinements within the porous layers and around the spherical boundary using the `SetLocalSizeOnShape` function. The `Reorient2DBy3D` function was also used to ensure that all boundary face normals pointed outwards of the enclosure. The step is crucial considering that the boundary condition is specified across these faces using expressions. The computed mesh is then be exported to file in `.unv` file format using function attributes of the mesh object. Conversion to `.xml.gz` for interfacing with the solver is achieved using FEConv [1] and `dolfin-convert` (provided by FEniCS) utilities as,

```
$ feconv -gm mesh.unv mesh.msh
$ dolfin-convert mesh.msh mesh.xml
$ gzip mesh.xml
```

A.1.2 Solver

The solver of choice is FEniCS, a popular open-source finite-element library for solving partial differential equations(PDEs). It offers a rich interface with data-structures and optimized algorithms for finite-element code which makes it easy to write PDEs. The library is hardware-optimized and parallel by design and is easily deployed to high-performance computing clusters. With its Python and C++ interfaces, FEniCS offers powerful capabilities to integrate into various scientific computing workflows.

The FEniCS library offers a number of component modules and the interfacing is done mainly through its DOLFIN and UFL modules. DOLFIN is a highly optimized computational back-end responsible for finite-element machinery and offers a rich Python interface. It provides abstract data-structures similar to mathematical terms such as mesh, finite element, function spaces and functions. It also includes compute-intensive algorithms such as finite-element assembly and mesh refinement, and, interfaces to various linear algebra solvers and libraries like PETSc and Eigen. UFL, on the other hand, provides an abstract mathematical language to express variational problems which are interpreted automatically and connected to DOLFIN classes.

The powerful feature of the solver is its ability to interpret the variational form in an easily readable framework. The Python module also allows for finer control through a detailed interface to the underlying C++ code enabling sub-classing and base class overloading. Among others, it provides an **Expression** class which can be used for user-defined expressions specified by C++ code and compiled during execution by a just-in-time (JIT) compiler for efficiency. A detailed documentation along with numerous examples are offered by Langtangen et al.[53] and at the official FEniCS documentation webpage[2]. It is to be noted that, currently the solver is limited by its inability to handle complex numbers (although it is under development, the release version did not contain it at the time of writing) and needs additional care to ensure that the real and imaginary parts of the equations and function spaces are represented separately.

The solver section handles details of the finite element implementation of the model and is made concise only due to the features of the FEniCS library. The geometrical parameters regarding the physical dimensions of the various subdomains and the model parameters related to frequency of choice, the user-defined boundary parameters etc are provided. Subsequently, the mesh is read into memory and each of the subdomains are marked by different identifiers, which is illustrated below.

```
### Fluid domain lengths : L[0], L[1], L[2] in x,y,z directions
### Porous domain radii : r_min (inner) and r_max (outer)
```



```

### Mesh object : mesh

# Obtain list of all cell identifiers
subdomains = MeshFunction("size_t", mesh, mesh.topology().dim())

# # Sub-domain specifications
tol = DOLFIN_EPS_LARGE

pml_cond = "fabs(x[0])>Lx-tol || fabs(x[1])>Ly-tol || fabs(x[2])>Lz-tol"
pml_layer = CompiledSubDomain(pml_cond,
                              Lx=L[0], Ly=L[1], Lz=L[2], tol=tol)

por_cond = "x.norm() > r_min-tol && x.norm() < r_max + tol"
por_layer = CompiledSubDomain(por_cond,
                              r_min=r_min, r_max=r_max, tol=tol)

# # Mark Sub-domains
subdomains.set_all(0)           # Fluid_Marker = 0
pml_layer.mark(subdomains, 1)   # PML_Marker = 1
por_layer.mark(subdomains, 2)   # Porous_Marker = 2

```

A similar procedure is also used to mark the boundary faces by different identifiers so as to specify boundary conditions. The function space is initialized with Raviart-Thomas finite elements, which define basis functions as unit vectors along the normals of the faces. To substitute a complex function space, it is necessary to initialize a hybrid function space made of two real parts,

```

# # Define function space (1st order Raviart-Thomas elements)
RT = d.FiniteElement("RT", mesh.ufl_cell(), 1)
Q = d.FunctionSpace(mesh, RT)

# # Define 2-part real function spaces to substitute for complex space
V = d.FunctionSpace(mesh, RT * RT)

(u_re, u_im) = d.TrialFunctions(V)
(v_re, v_im) = d.TestFunctions(V)

```

The variational form maybe expressed in three separate stages - one each for the fluid, porous and PML subdomains. The primary task is to represent each term on the variational equation in real and imaginary portions. The wave equations being linear easily separate into two terms. For example consider the variational formulation of a porous coupled model i.e

$$\begin{aligned}
& \int_{\Omega_F} \rho_F c^2 \Re(\operatorname{div} \mathbf{u}) \Re(\operatorname{div} \mathbf{v}) \, dV - \int_{\Omega_F} \rho_F \omega^2 \Re(\mathbf{u}) \cdot \Re(\mathbf{v}) \, dV \\
& \int_{\Omega_F} \rho_F c^2 \Im(\operatorname{div} \mathbf{u}) \Im(\operatorname{div} \mathbf{v}) \, dV - \int_{\Omega_F} \rho_F \omega^2 \Im(\mathbf{u}) \cdot \Im(\mathbf{v}) \, dV \\
& + \int_{\Omega_P} \Re(K_P(\omega) \operatorname{div} \mathbf{u}) \Re(\operatorname{div} \mathbf{v}) \, dV - \int_{\Omega_P} \Re(\rho_P(\omega) \omega^2 \mathbf{u}) \cdot \Re(\mathbf{v}) \, dV \\
& + \int_{\Omega_P} \Im(K_P(\omega) \operatorname{div} \mathbf{u}) \Im(\operatorname{div} \mathbf{v}) \, dV - \int_{\Omega_P} \Im(\rho_P(\omega) \omega^2 \mathbf{u}) \cdot \Im(\mathbf{v}) \, dV \\
& + \int_{\Omega_{\text{PML}}} \rho_F c^2 \Re(\widetilde{\mathcal{C}}(\nabla \mathbf{u})) : \Re(\nabla \mathbf{v}) \, dV - \int_{\Omega_{\text{PML}}} \rho_F \omega^2 \Re(\widetilde{\mathbf{M}} \mathbf{u}) \cdot \Re(\mathbf{v}) \, dV \\
& + \int_{\Omega_{\text{PML}}} \rho_F c^2 \Im(\widetilde{\mathcal{C}}(\nabla \mathbf{u})) : \Im(\nabla \mathbf{v}) \, dV - \int_{\Omega_{\text{PML}}} \rho_F \omega^2 \Im(\widetilde{\mathbf{M}} \mathbf{u}) \cdot \Im(\mathbf{v}) \, dV = 0. \quad (\text{A.1.1})
\end{aligned}$$

This equation can be easily specified in FEniCS using UFL. Since the subdomains are marked with identifiers already, the bilinear form within the fluid layer maybe specified directly as,

```

# FLUID LAYER : SubDomain Marker 0
# Define Bilinear form within the fluid layer
a = (rho * c**2 * div(u_re) * div(v_re) * dx(0)
      + rho * c**2 * div(u_im) * div(v_im) * dx(0)
      - rho * omega**2 * inner(u_re, v_re) * dx(0)
      - rho * omega**2 * inner(u_im, v_im) * dx(0))

```

The PML and the porous layer require operating between complex arithmetic and UFL operations. These are made simpler by declaring a complex container class with the necessary operations like multiplication, division and conjugation. This reduces the complex arithmetic into UFL operations and increases readability of the code. The `Complex` class declared below shows a simple structure of such a container class which contains member functions to return just the real or imaginary parts of the common operations. The return values of each of them translates the arithmetic into UFL operations.

```

class Complex(object):
    def __init__(self, real_part, imag_part):
        self.real=real_part
        self.imag=imag_part

    # Define inner-(conjugated) product
    def dot_re(f, g):
        return f.real*g.real + f.imag*g.imag

```

```

def dot_im(f, g):
    return f.imag*g.real - f.real*g.imag

# Define product
def prod_re(f, g):
    return f.real*g.real - f.imag*g.imag

def prod_im(f, g):
    return f.imag*g.real + f.real*g.imag

# Define division
def div_re(f, g):
    return dot_re(f, g)/dot_re(g, g)

def div_im(f, g):
    return dot_im(f, g)/dot_re(g, g)

```

Further consideration is needed to specify piece-wise functions (γ_j 's) into the variational form. A direct approach would be segregating the PML domain into smaller domains defined by intervals where γ_j 's are resolved, and specify the operations separately. While this approach saves the assembling time, it can quickly get tedious. Instead, they can be easily expressed using logical operators offered by UFL [5], specifically the `conditional` operator. This leaves the evaluations to be performed by the assembler during the assembling phase. The following code illustrates the bilinear form expression using the `Complex` class and the UFL logical operators. The real and imaginary parts of the tensor product $\tilde{\mathbf{C}}(\nabla \mathbf{u}) : \nabla \mathbf{v}$ and the inner product $\tilde{\mathbf{M}} \mathbf{u} \cdot \mathbf{v}$ are expressed by simplifying the operation.

```

# PML LAYER : SubDomain Marker 1
def s(j):
    return conditional(gt(abs(x[j]), L[j] + tol),
                      c / abs(abs(x[j]) - (L[j] + Lpml)) / omega,
                      Constant("0."))

def gamma(j):
    return Complex(Constant("1.0"), s(j))

def du_dx(u_re, u_im, i):
    return Complex(Dx(ure[i], i), Dx(uim[i], i))

# # Divergence operator in the PML domain
Div_re = sum(prod_re(Complex(1., 0.) / gamma(i), du_dx(u_re, u_im, i))
             for i in range(3))
Div_im = sum(prod_im(Complex(1., 0.) / gamma(i), du_dx(u_re, u_im, i))
             for i in range(3))

# # Scaled PML displacement vector to be used in the mass matrix
def coef(ure, uim, i)
    return Complex(ure[i], uim[i])

```

```

u_scaled_re = as_vector([prod_re(gamma(i), coef(u_re, u_im, i))
                        for i in range(3)])
u_scaled_im = as_vector([prod_im(gamma(i), coef(u_re, u_im, i))
                        for i in range(3)])

dx_pml = dx(1, scheme='default', degree=6)

## Define bilinear form in the PML layer
a += (rho * c**2 * Div_re * d.div(v_re) * dx_pml
      + rho * c**2 * Div_im * d.div(v_im) * dx_pml
      - rho * omega2 * inner(u_scaled_re, v_re) * dx_pml
      - rho * omega2 * inner(u_scaled_im, v_im) * dx_pml)

```

The expression for porous layer requires the porous density and bulk modulus obtained from the JCAL model. These are easily computed from expressions (3.2.4) and (3.2.5) using Python's inbuilt complex arithmetic support and the results are subsequently cast into `Complex` class type. The resulting bilinear expressions can then be specified as,

```

# POROUS LAYER : SubDomain Marker 2
div_u = Complex(div(u_re), div(u_im))
a += (prod_re(BulkMod_P, div_u) * div(v_re) * dx(2)
      + prod_im(BulkMod_P, div_u) * div(v_im) * dx(2)
      - omega**2 * inner(prod_re(Rho_P, u), v_re) * dx(2)
      - omega**2 * inner(prod_im(Rho_P, u), v_im) * dx(2))

```

The boundary conditions at the structure are also divided into real and imaginary parts and the expressions are specified using the `Expression` class. They can then be applied over the relevant faces using the `DirichletBC` class.

```

# BC at Structure Boundary (Marker 2)
bc = [DirichletBC(V.sub(0), g_re, bdry_markers, 2), # Real subspace
      DirichletBC(V.sub(1), g_im, bdry_markers, 2)] # Imag subspace

```

The system is now completely determined and can be assembled and solved by the MUMPS (MULTifrontal Massively Parallel Sparse direct Solver) linear algebra solver by using DOLFIN provided `solver` interface. The resulting solution is separated into real and imaginary parts using the `split` attribute of the `Function` class and are then written into XDMF files using the DOLFIN provided `XDMFFile` class.

A.1.3 Post-processing and Visualization

Visualization of generated simulation data is critical in understanding the physical process. Implementing and representing this data in a simple and effective manner is extremely

useful for deriving information, presenting results, and also in testing and debugging. The post-processing operations on the solution is dependent on the study undertaken by the user. In this specific use case, some routine cases of analysis include validation of the solver for a test case, measure of a field at a particular point in space and directivity patterns of fields around the object amongst many others. The implementation of these could either be included in the solver phase or during the visualization phase. Within the solver phase, these could just be operations on the solution data done using Python and plotted using some common user preferred graphing libraries like Matplotlib[49]. The approach quickly gets overwhelming while dealing with 3D datasets and it is useful to use a dedicated visualization tool like ParaView. It is a widely used open-source visualization tool for plotting and viewing solutions and graphs. It offers a powerful and an intuitive 3D visualization interface allowing for heavy in-situ customization and processing of simulation data. Furthermore, it also provides a Python scripting interface to automate visualization for batch processing.

ParaView uses a three-stage procedure for visualization of data: reading, filtering and rendering, all done using the user interface. The simulation data from the solver is read into memory through many supported file formats. The dataset being typically large, the XDMF (eXtensible Data Model and Format) file format is used for storage, which is able to manage extremely large datasets and is scalable for parallel systems. Filters provide the ability to extract or analyze this data into information. There are a wide range of filters available for analysis and visualization including plotting graphs, contours, surface plots, vector field plots etc. In addition, it is also possible to define user-defined filters to perform customized operations. The rendering stage deals with generating images or interactive plots from the filtered information. ParaView provides a user guide[9] and many tutorials[3] highlighting the usage and relevance of each of the stages along with the available functionalities to fully exploit its potential.

The XDMF file exported from FEniCS are directly compatible and can be easily visualized using ParaView. The user interface is very intuitive and once a file is opened it allows the user to select the solution fields to import into memory. Once the datasets are imported, it renders the volumetric data on the viewer. The toolbar offers some commonly used filters and are also accessible from the menu options. Initially, the dataset is bifurcated into truncated fluid domain and the PML. This can be achieved using the `ExtractCellsByRegion` filter used with its 'box' configuration scaled appropriately to omit the cells in the PML. To obtain cross-sections of the volumetric data, as shown in Fig. A.3, the filter `Clip` or `Slice` is used and the specifications of the cutting plane is provided. It is also possible to compute from the saved fields on the interface directly by using the filter `Calculator`. This filter allows the user to define an expression using the fields in memory and compute a derived field. It is useful in computing the total displacement field putting together the real and imaginary portions. The same filter can also be used to compute the errors provided the exact solution is also in memory. ParaView offers a lot more features besides the one elaborated here e.g plotting, thresholding etc. which can help the user gain further understanding from the simulation data. Certain computations like pressure-at-a-point and

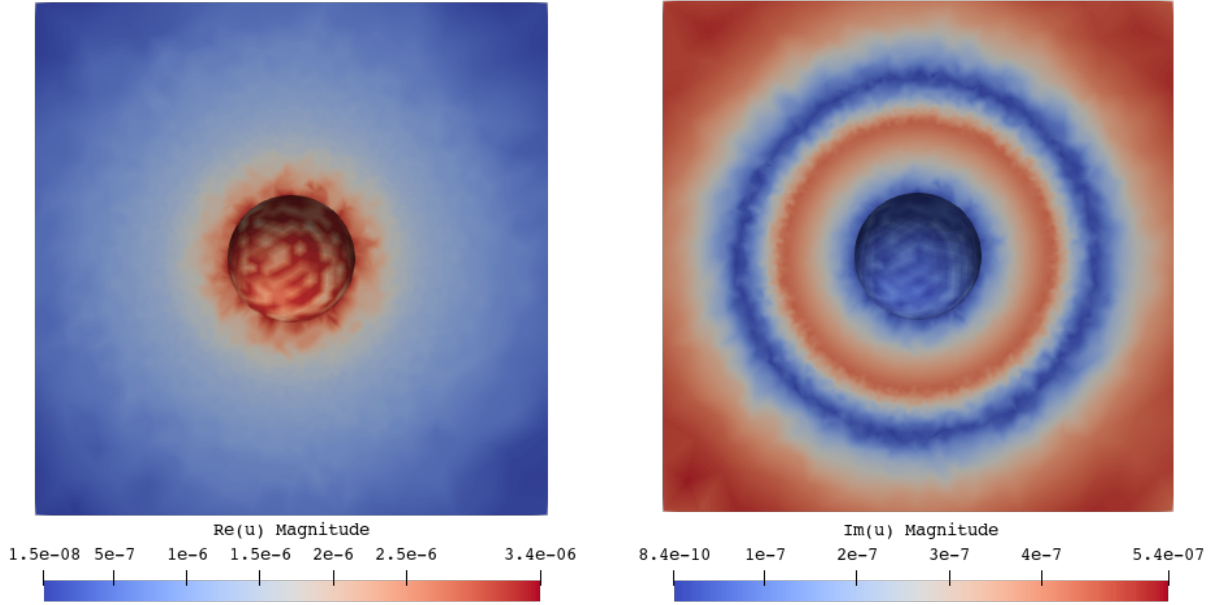


Figure A.3: Cross-sectional contour views of the magnitude of the real part (left) and the magnitude of the imaginary part (right) of the computed displacement field computed using Clip filter of ParaView.

directivity computations are better handled in the solver stage and can then be visualized using other Python graphing libraries.

A.1.4 Github Repository

Latest release: <https://doi.org/10.5281/zenodo.5171815>

GitHub repository: <https://github.com/ROMSOC/benchmarks-acoustic-propagation>

The project github repository contains scripts for generating the relevant meshes and computing the scattering of a plane wave by the Microflown PU Regular Probe and other simple geometries such as a three-dimensional spheres. More precisely, the repository includes all the data and Python scripts to run from scratch the following three coupled problems:

- A three-dimensional coupled fluid-PML acoustic scattering problem with an spherical obstacle.
- A three-dimensional coupled fluid-porous-PML acoustic scattering problem with an spherical obstacle coated with a spherical porous layer.
- A three-dimensional coupled fluid-porous-PML acoustic scattering problem, where the obstacle is a realistic design of a PU probe coated with a cylindrical porous layer.

These three benchmarks share a common mathematical model and an analogous variational formulation. All of them have been discretized with the same kind of finite element method.

The repository contains three different scripts to build the meshes (in folder `source/00_mesher`), which corresponds to the three different benchmarks included in the repository:

```
$ salome -t microflown_pu_probe.py # Mesh for coupled PU probe
$ salome -t sphere_PML.py # Mesh for coupled fluid-PML
$ salome -t sphere_porous_PML.py # Mesh for coupled fluid-porous-PML
```

The Python scripts related with the three problems are structured in two folders within the GitHub repository :

- Folder `source/02_scattering_sphere/`: it contains the planewave scattering benchmarks involving spherical geometries. This directory contains scripts to develop and analyze FEM method to solve for scattering by a sphere of an incident plane/radial wave. Since the exact solutions can be obtained for this scenario it is ideal to perform analysis of the FEM method and also validate the solver. This folder contains the first two benchmarks regarding spherical geometries:
 - Subfolder `source/02_scattering_sphere/planewave_scattering/`: it contains the planewave scattering benchmark involving only rigid sphere surrounded by fluid. In this benchmark, an incident plane wave is impinging on a rigid sphere, and these scripts computes the scattered field utilizing a classical quadratic PML absorption boundary condition. The variational form is written in terms of the scattered field. Exact solution is known and the errors can be computed to validate the solver. The absorption coefficients for the quadratic PML functions are estimated using the monopole test case and used here. Exact solution is provided by C++ scripts provided in the exact folder which are compiled during runtime by the FEniCS provided JIT compiler. FE Errors can be computed and convergence behaviour can be investigated using these scripts. To run the numerical simulation, just type the Python script:

```
# Solver for the coupled fluid-PML
$ python3 scattering_sphere_classical_PML.py
```

- Subfolder `source/02_scattering_sphere/radialwave_scattering_porous_coupling/` it contains the radial wave scattering by a rigid sphere with a spherical porous layer. A radial wave is incident on a rigid sphere which is surrounded by a spherical porous layer. The solver computes the resulting field using the Johnson-Champoux-Allard-Lafarge (JCAL) model for a few typical parameter values, and compares them against the exact solution. The exact solution requires an inverse method to compute radial wave amplitudes within the subdomains (see a

detailed description of this benchmark in [62, Chapter 2]). To run the numerical simulation, the following Python script must be used:

```
# Solver for the coupled fluid-porous-PML
$ python3 scattering_sphere_porous.py
```

- Folder `source/03_scattering_pu_probe/`: it contains the planewave scattering benchmark involving a Microflow PU Probe. In this benchmark, an incident plane wave is scattered by the Microflow PU Probe which may contain a steel mesh (treated as acoustically equivalent to micro-perforated panels) and a porous windscreen. Dah-You-Maa's model is used for the MPP and JCAL model for the porous layer. Optimal PML model is used to replicate the non-reflecting boundaries. The ensemble is coupled with interface conditions and solved using the displacement formulation of the Helmholtz-like equations using the FEM method. Notice that the parameters for these models need to be obtained from experimental data. To run the solver, a Python script should be executed:

```
# Solver for the PU probe
$ python3 scattering.py
```


Summary

Sound intensity PU probes are capable of measuring both sound pressure as well as acoustic particle velocity. Although they are well suited for most indoor applications, it is known that they exhibit high sensitivity to air flow. For their effective use in outdoor conditions, porous windscreens are often utilized to mitigate flow-induced noise. This thesis proposes a rigorous mathematical framework to model the influence of (i) the probe packaging and design, (ii) the steel mesh envelopes around the probe, (iii) the porous windscreens, (iv) laminar flow on both sensors.

A hierarchy of acoustic models are chosen for each medium - the steel meshes are modeled using Maa's model; the porous medium is considered acoustically as equivalent fluid with the Johnson-Champoux-Allard-Lafarge model accounting for its dissipative effects and the influence of an underlying laminar flow on the acoustic fluid is modeled using the augmented Galbrun's model formulation. All of the mathematical models consider a time-harmonic regime and are formulated using the displacement field as the primal unknown. The suggested approach simplifies the interfacing between the models of different media. The models are implemented and validated with test cases using the finite element method to solve the arising differential equations in each context. Furthermore, detailed derivations are provided for coupling these individual models.

Numerical results are demonstrated for a sound intensity PU probe along with its windscreens for the case of no flow and laminar flow. A solver is developed for measuring the package gain of a probe housing, the insertion loss of a porous windscreen, and predicting the acoustic impact of laminar flow. The numerical solvers are implemented entirely using open-source software and details on the implementation are provided along with the code. Distinctly, the computational models are also complemented with an experimental investigation performed especially for characterizing the physical parameters of porous media.

Finally, a Darcy-Forcheimer-Galbrun model is proposed and rigorously developed from the basic principles of continuum mechanics to model the acoustic influence of porous medium particularly in presence of an underlying flow. With these coupled ensembles the study aims to initiate the development of a digital twin of sound intensity PU probes and predict the acoustical influence for a given configuration of porous windscreen and airflow.

Resumen en castellano

Las sondas de intensidad PU son capaces de medir tanto la presión sonora como la velocidad de partícula. A pesar que son compatibles con la mayoría de aplicaciones de interior, los sensores pueden presentar problemas si hay flujo de aire presente en el entorno de medida. Para su uso en aplicaciones de exteriores, materiales porosos son utilizados para mitigar el ruido inducido por el flujo de aire cercano a los transductores. Esta tesis propone un marco matemático riguroso para modelar la influencia de (i) el diseño y empaque de la sonda, (ii) posibles mallas de acero alrededor de la sonda, (iii) protectores contra el viento de material poroso, (iv) el flujo laminar en ambos sensores.

Una serie de modelos acústicos fueron seleccionadas para la simulación de cada medio: las mallas de acero se modelan utilizando el modelo de Maa; el medio poroso se considera como un fluido equivalente con el modelo de Johnson-Champoux-Allard-Lafarge que tiene en cuenta sus efectos disipativos; y la influencia de un flujo laminar subyacente en el fluido acústico se modela utilizando la formulación del modelo de Galbrun aumentado. Todos los modelos matemáticos consideran un régimen armónico temporal y se formulan utilizando el campo de desplazamiento como la incógnita primaria. El enfoque sugerido simplifica la interconexión entre los modelos de diferentes medios. Los modelos fueron implementados utilizando el método de elementos finitos para resolver las ecuaciones diferenciales que surgen en cada contexto. Una serie de simulaciones numéricas y experimentales fueron llevadas a cabo para validar la metodología propuesta. Además, se proporcionan detalladas derivaciones para acoplar múltiples modelos de forma individual.

Una serie de simulaciones numéricas fueron realizadas para caracterizar la respuesta de una sonda de intensidad sonora PU junto con un elemento protector de viento, tanto para el caso de flujo laminar como sin flujo. Herramientas especializadas para medir la ganancia introducida por el diseño de la sonda, la pérdida de inserción de un protector poroso y estimar el impacto de un flujo laminar en las respuestas de los sensores. Las soluciones numéricas se implementaron en su totalidad utilizando software de código abierto y los detalles sobre la implementación se proporcionan junto con el código. Asimismo, los modelos computacionales también se complementan con una investigación experimental realizada especialmente para caracterizar los parámetros físicos de medios porosos.

Finalmente, se propone un modelo Darcy-Forscheimer-Galbrun y se desarrolla rigurosamente a partir de los principios básicos de la mecánica continua para modelar la influencia acústica del medio poroso, particularmente en presencia de un flujo subyacente. Con estos conjuntos acoplados, el estudio tiene como objetivo iniciar el desarrollo de un gemelo digi-

tal (digital-twin) de sondas de intensidad para poder predecir la influencia acústica en una configuración dada incluyendo la presencia de flujo de aire.

Further research

The work reported in this thesis opens the door to many questions and possible further improvements of the proposed methods. Some suggestions for future research, and topics that have not been investigated, are listed below:

- While the acoustic behaviour of submillimeter steel meshes were investigated in this work, a welcome inclusion would be to include models for characterizing the acoustic behavior of thin flexible materials like cloth, often used in complex multi-layer windscreens.
- The current work reported the prediction of acoustic performance of a single-layer foam windscreen with a steel mesh. The investigation could be expanded to multi-layered general porous materials e.g, fibrous porous materials.
- A numerical framework was proposed is focused on the simulation and validation of a specific probe and windscreen geometry. An extension of this study is to develop inverse characterization and shape optimization to cater to a specific application.
- All the methodologies suggested here based on FEM strategies, which to some extent limits the maximum frequency due to computational limitations. Alternative methodologies should be investigated to extend the simulation frequency range, and cope with a wider acoustic spectrum.
- All of the proposed models would gain more reliability under a definitive comparison against thorough experimental results.

Fundings and Support

- This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Grant Agreement No. 765374.
- Support from ED431C 201833 - M2NICA research group (Xunta de Galicia & ERDF) and ED431G 201901 - CITIC (Xunta de Galicia & ERDF).
- CESGA for access to the Finnis-Terrae supercomputer.

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